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PROPERTIES OF THE
LINEARIZED KEPLER OPERATOR

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Abstract

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For N≥3, there are no explicit general solutions to the gravitational motion of N bodies under the laws of classical Newtonian physics, besides those generated by the central configurations (discovered for N=3 by Euler and Lagrange). For N=2, a complete classification of the solutions is well-known; it reduces to the Kepler problem, the motion of a single body under a central gravitational force. Investigations of N-body solutions typically use perturbation methods to show the existence of exact solutions. These techniques start with inexact solutions, based on the exact solutions to the Kepler problem, which are then perturbed until they converge to an exact solution.

We investigate the mathematical properties of the linearized Kepler operator, using methods of ODE theory and functional analysis. We calculate the complete set of solutions to the linearized Kepler operator, for the case in which the initial Kepler solutions are hyperbolas. We show that exactly one of these solutions is bounded. We show that the linearized Kepler operator is a Fredholm operator on weighted spaces of continuously differentiable functions, and we calculate its index and the dimensions of its kernel and cokernel. We obtain a further result that relates the case in which the linearized Kepler operator is surjective, to a more useful case in which it is not surjective, by means of a functional decomposition.

1. Introduction

A mathematical problem of great interest, due to its mathematical difficulty as well as its practical application to astronomy, is the gravitational motion of N bodies (considered as point masses), under the laws of classical Newtonian physics. This is known as the *N-body problem.* Indeed, it can be argued that Poincaré's initial discoveries that led to the development of the important mathematical field of study now known as algebraic topology, were a by-product of his investigation of the N-body problem [**12**].

For $N = 2$, a complete classification of the solutions is well-known. For two bodies, the problem readily reduces to the *Kepler problem*, which is the motion of a single body under the influence of a central gravitational force. All the solutions follow "orbits" that are conic sections: straight lines, circles, ellipses, parabolas, and hyperbolas. But for $N \geq 3$, exact solutions are known for only a small number of special cases. More general solutions to the N-body problem have defied the efforts of four centuries of mathematicians and physicists.

When $N = 2$, the energy of the system is positive, and solutions involving collisions are excluded, the trajectories of both bodies are hyperbolas. Consequently, in this case, both bodies have "orbits" that are asymptotic to straight lines as time $t \to -\infty$ and as time $t \to +\infty$. Furthermore, both bodies tend toward constant, non-zero velocity at $t = \pm\infty$. A more general result, due to Jean Chazy [**4**], can be paraphrased as follows:

Chazy Theorem: Given a Newtonian gravitational system of N bodies with masses $m_1, m_2, \ldots, m_N$, with $\mathbb{R}^d$ as the coordinate space of each body. Suppose that a solution $\mathbf{x}(t) \in \mathbb{R}^{dN}$ has the properties that each body's velocity $\dot{\mathbf{x}}_i(t)$ tends to a non-zero limit as $t \to +\infty$, and simultaneously, the distances $r_{ij}(t) = \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|$ between any two of the bodies tends to infinity. Then, as $t \to +\infty$, the solution has an asymptotic expansion of the form

$$\mathbf{x}_i(t) = \mathbf{A}_i^+ t + \mathbf{B}_i^+ \log t + \mathbf{C}_i^+ + \mathcal{O}\left(\frac{\log t}{t}\right),$$

where $\mathbf{A}_i^+, \mathbf{B}_i^+, \mathbf{C}_i^+$ are constant vectors in $\mathbb{R}^d$. Similarly, suppose that a solution $\mathbf{x}(t) \in \mathbb{R}^{dN}$ has the properties that each body's velocity $\dot{\mathbf{x}}_i(t)$ tends to a non-zero limit as $t \to -\infty$, and simultaneously, the distances $r_{ij}(t) = \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|$ between any two of the bodies tends to infinity. Then, as $t \to -\infty$, the solution has an asymptotic expansion of the form

$$\mathbf{x}_i(t) = \mathbf{A}_i^- t + \mathbf{B}_i^- \log t + \mathbf{C}_i^- + \mathcal{O}\left(\frac{\log t}{t}\right),$$

where $\mathbf{A}_i^-, \mathbf{B}_i^-, \mathbf{C}_i^-$ are constant vectors in $\mathbb{R}^d$.

Objective: The objective of our present research is to develop some mathematical tools that will be useful for the further investigation of N-body solutions that exhibit the *hyperbolic escape* properties described in the Chazy Theorem. Many investigations of N-body solutions use perturbation methods to show the existence of exact solutions. These techniques start with postulated, inexact solutions, which are then perturbed until they converge to an exact solution. Typically, the inexact solutions are based on the known exact solutions to the

Kepler problem, ignoring the additional "coupling" that occurs when more than two bodies interact. Then the "coupling" is introduced as perturbations to the inexact solutions.

In this research, we investigate the mathematical properties of the "Kepler operator", and in particular the *linearized Kepler operator*. We employ techniques from ODE theory and functional analysis to characterize the behavior of the linearized Kepler operator under perturbations of the initial value parameters of exact Kepler solutions. We concentrate on the case in which the initial Kepler solutions are hyperbolas.

2. Hamiltonian of the Three-Body Problem

We begin by considering the three-body problem in the plane, under the laws of classical Newtonian physics. Our three bodies have masses m_1, m_2, and m_3. Take $\mathbf{q}_1 \in \mathbb{R}^2$, $\mathbf{q}_2 \in \mathbb{R}^2$, and $\mathbf{q}_3 \in \mathbb{R}^2$ to be the Cartesian coordinates of the three bodies, with conjugate momentum variables $\mathbf{p}_1 \in \mathbb{R}^2$, $\mathbf{p}_2 \in \mathbb{R}^2$, and $\mathbf{p}_3 \in \mathbb{R}^2$. We assume center of mass coordinates, so that:

$$\sum_{j=1}^{3} m_j \mathbf{q}_j = 0\,. \tag{1}$$

We also take the total linear momentum to be 0:

$$\sum_{j=1}^{3} \mathbf{p}_j = 0\,. \tag{2}$$

Now consider Jacobi coordinates $\mathbf{x}_1 \in \mathbb{R}^2$ and $\mathbf{x}_2 \in \mathbb{R}^2$:

$$\mathbf{x}_1 = \mathbf{q}_2 - \mathbf{q}_1 \tag{3}$$

$$\mathbf{x}_2 = \mathbf{q}_3 - \frac{m_1\mathbf{q}_1 + m_2\mathbf{q}_2}{m_1 + m_2}, \tag{4}$$

with corresponding conjugate momentum variables $\mathbf{y}_1 \in \mathbb{R}^2$ and $\mathbf{y}_2 \in \mathbb{R}^2$ (there are two more variables in Jacobi coordinates, but they are the position of the center of mass and the total linear momentum and hence are both 0). Note that, in accordance with the notation in Meyer **[10]**, $\mathbf{p}_i$ and $\mathbf{q}_i$ are used for position and momentum in Cartesian coordinates, whereas $\mathbf{x}_i$ and $\mathbf{y}_i$ are used for position and momentum in Jacobi coordinates.

In the Jacobi coordinates given above, the Hamiltonian of the three-body problem is:

$$H = \frac{\|\mathbf{y}_1\|^2}{2M_1} + \frac{\|\mathbf{y}_2\|^2}{2M_2} - \frac{m_1 m_2}{\|\mathbf{x}_1\|} - \frac{m_2 m_3}{\|\mathbf{x}_2 - \alpha_1 \mathbf{x}_1\|} - \frac{m_1 m_3}{\|\mathbf{x}_2 + \alpha_2 \mathbf{x}_1\|}, \tag{5}$$

where the gravitational constant G has been taken to be unity and:

$$M_1 = \frac{m_1 m_2}{m_1 + m_2}, \qquad M_2 = \frac{(m_1 + m_2)\, m_3}{m_1 + m_2 + m_3}, \tag{6}$$

$$\alpha_1 = \frac{m_1}{m_1 + m_2}, \qquad \alpha_2 = \frac{m_2}{m_1 + m_2}\,. \tag{7}$$

(See **[10]**, page 29. Our notation differs from Meyer's notation in that we are indexing the m's and the α's from 1 rather than from 0.)

Now we re-write the above Hamiltonian (equation (5)) as

$$\begin{aligned} H &= \frac{\|\mathbf{y}_1\|^2}{2M_1} - \frac{(m_1 + m_2)\, M_1}{\|\mathbf{x}_1\|} \\ &\quad + \frac{\|\mathbf{y}_2\|^2}{2M_2} - \frac{(m_1 + m_2 + m_3)\, M_2}{\|\mathbf{x}_2\|} \\ &\quad + \frac{(m_1 + m_2)\, m_3}{\|\mathbf{x}_2\|} - \frac{m_1 m_3}{\|\mathbf{x}_2 + \alpha_2 \mathbf{x}_1\|} - \frac{m_2 m_3}{\|\mathbf{x}_2 - \alpha_1 \mathbf{x}_1\|} \\ &= H_1 + H_2 + H_c, \end{aligned} \tag{8}$$

where

$$H_1 = \frac{\|\mathbf{y}_1\|^2}{2M_1} - \frac{(m_1 + m_2)\, M_1}{\|\mathbf{x}_1\|}, \tag{9}$$

$$H_2 = \frac{\|\mathbf{y}_2\|^2}{2M_2} - \frac{(m_1 + m_2 + m_3)\, M_2}{\|\mathbf{x}_2\|} \tag{10}$$

$$H_c = m_1 m_3 \left(\frac{1}{\|\mathbf{x}_2\|} - \frac{1}{\|\mathbf{x}_2 + \alpha_2 \mathbf{x}_1\|} \right) + m_2 m_3 \left(\frac{1}{\|\mathbf{x}_2\|} - \frac{1}{\|\mathbf{x}_2 - \alpha_1 \mathbf{x}_1\|} \right) \tag{11}$$

Note that H_1 and H_2 each have the form of the Hamiltonian for a *central force Kepler problem* for each of the two Jacobi vectors $\mathbf{x}_1$ and $\mathbf{x}_2$. We will investigate the Kepler problem in more detail in the next section. The term H_c represents the *coupling* between the two Kepler problems.

3. The Kepler Problem

We now consider the case of the gravitational central force problem in the plane, known as the Kepler problem. As is well-known, the same transformation to Jacobi coordinates given in the preceding section (but with only two bodies and hence no vectors $\mathbf{x}_2$ and $\mathbf{y}_2$) reduces the two-body problem to the central force problem. Hence the Cartesian coordinates for the central force problem are in fact the Jacobi coordinates for the related two-body problem, and we shall denote them by $\mathbf{x}$ (position) and $\mathbf{y}$ (momentum). In these coordinates, the Hamiltonian for the Kepler problem is:

$$H = \frac{\|\mathbf{y}\|^2}{2m} - \frac{Mm}{\|\mathbf{x}\|}, \tag{12}$$

The above formulation is for a particle of mass m moving under the influence of a central force due to a "stationary" mass M; the gravitational constant G has been taken to be unity, or (equivalently) G is absorbed into the central mass $M = Gm_0$. Furthermore, the above Hamiltonian formulation applies to the *reduced* two-body problem (or to the first Jacobi vector in the three-body problem, ignoring coupling) if we simply take M to be $G(m_1 + m_2)$ and take m to be $M_1 = m_1 m_2 / (m_1 + m_2)$. The above Hamiltonian formulation also applies to the second Jacobi vector in the three-body problem (again, ignoring coupling) if we simply take M to be $G(m_1 + m_2 + m_3)$ and take m to be $M_2 = (m_1 + m_2) m_3 / (m_1 + m_2 + m_3)$.

Let J be the standard symplectic 2-form and J^* be the corresponding isomorphism from the dual space of $\mathbb{R}^4$ onto $\mathbb{R}^4$. If we let $\mathbf{z} = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, then the Hamiltonian differential equation is $\dot{\mathbf{z}} = J^* \left(dH \left(\mathbf{z} \right) \right)$. In matrix form, this is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} dH \left(x_1, x_2, y_1, y_2 \right), \tag{13}$$

where $dH \left(x_1, x_2, y_1, y_2 \right)$ is written as a column vector (for the purpose of matrix multiplication). For the Kepler problem, in Cartesian coordinates, this works out to be the Cartesian form of Newton's equations of motion:

$$\dot{x}_1 = \frac{y_1}{m} \tag{14}$$

$$\dot{x}_2 = \frac{y_2}{m} \tag{15}$$

$$\dot{y}_1 = -\frac{Mm x_1}{\left\| \mathbf{x} \right\|^3} \tag{16}$$

$$\dot{y}_2 = -\frac{Mm x_2}{\left\| \mathbf{x} \right\|^3}. \tag{17}$$

For the Kepler problem in the plane, the Hamiltonian in polar coordinates is given by:

$$H \left(r, \theta, R, \Theta \right) = \frac{1}{2m} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{Mm}{r}, \tag{18}$$

where the coordinates are r, θ, and their symplectic conjugates R, Θ. (Note that H does not depend on θ, since the problem is "rotationally invariant".)

Again let J be the standard symplectic 2-form and J^* be the corresponding isomorphism from the dual space of $\mathbb{R}^4$ onto $\mathbb{R}^4$. If we let $z = (r, \theta, R, \Theta)$, then the Hamiltonian differential equation is $\dot{\mathbf{z}} = J^*(dH(\mathbf{z}))$. In matrix form, this is

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{R} \\ \dot{\Theta} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} dH(r, \theta, R, \Theta), \tag{19}$$

where $dH(r, \theta, R, \Theta)$ is written as a column vector (for the purpose of matrix multiplication). For the Kepler problem, in polar coordinates, this works out to be the polar form of Newton's equations of motion:

$$\dot{r} = \frac{R}{m} \tag{20}$$

$$\dot{\theta} = \frac{\Theta}{mr^2} \tag{21}$$

$$\dot{R} = \frac{\Theta^2}{mr^3} - \frac{Mm}{r^2} \tag{22}$$

$$\dot{\Theta} = 0. \tag{23}$$

4. Parameters of Kepler Orbits

The general solution of the Kepler ordinary differential equation (ODE) system is a function of t, $\widetilde{r}$, $\widetilde{\theta}$, $\widetilde{R}$, and $\widetilde{\Theta}$, where t denotes time and $\widetilde{r}$, $\widetilde{\theta}$, $\widetilde{R}$, and $\widetilde{\Theta}$ are the initial values of r, θ, R, and Θ at $t = 0$. Note that

$$\dot{H} = dH(\mathbf{z})(\dot{\mathbf{z}}) = dH(\mathbf{z})(J^*(dH(\mathbf{z}))) = J(dH(\mathbf{z}), dH(\mathbf{z})) = 0, \tag{24}$$

since J is an alternating 2-form. Thus the energy H is constant on solutions, so we have

$$H = \widetilde{H} = \frac{1}{2m}\left(\widetilde{R}^2 + \frac{\widetilde{\Theta}^2}{\widetilde{r}^2}\right) - \frac{Mm}{\widetilde{r}}\,. \tag{25}$$

Furthermore, $\Theta = mr^2\dot{\theta}$ is simply the angular momentum, and it is constant on solutions, since $\dot{\Theta} = 0$, so we also have $\Theta = \widetilde{\Theta}$.

As is well-known, for a given set of initial values, the general solution traverses a conic section. Excluding the case $\widetilde{\Theta} = 0$ (for which the solutions are straight lines that pass through the origin, *i.e.*, collisions or ejections), each such conic section satisfies the following equation, in polar coordinates (see **[13]**, page 8):

$$r = \frac{\widetilde{\Theta}^2/Mm^2}{1 + e\cos(\theta - g)} = \frac{\widetilde{\Theta}^2/Mm^2}{1 + e\cos f}, \tag{26}$$

where e is a non-negative constant (called the *eccentricity*) and g is the angle of inclination between the x-axis and the conic's major axis, its axis of symmetry (g is called the *argument of the perihelion*). The angle $f = \theta - g$ is known as the *true anomaly*. Now if we let $k = \widetilde{\Theta}^2/Mm^2$, we can write equation (26) above as

$$r = \frac{k}{1 + e\cos(\theta - g)} \,. \tag{27}$$

Differentiating equation (27) above, using the notation $\dot{r} = dr/dt$ and $\dot{\theta} = d\theta/dt$, we get:

$$\begin{aligned} \dot{r} &= \frac{-k}{(1 + e\cos(\theta - g))^2} \left(-e\sin(\theta - g)\right)\dot{\theta} \\ &= \frac{r^2\dot{\theta}e}{k} \sin(\theta - g) \,. \end{aligned} \tag{28}$$

From the equations of motion, we know that $\dot{r} = R/m$ and $r^2\dot{\theta} = \widetilde{\Theta}/m$. Substituting these expressions into equation (28) above, we get:

$$\begin{aligned} R &= \frac{\widetilde{\Theta}e}{k} \sin(\theta - g) \\ &= \frac{Mm^2e}{\widetilde{\Theta}} \sin(\theta - g) \,. \end{aligned} \tag{29}$$

In particular, the equations for r and R in terms of and θ must hold at $t = 0$. Therefore:

$$\widetilde{r} = \frac{k}{1 + e\cos\left(\widetilde{\theta} - g\right)}, \tag{30}$$

$$\widetilde{R} = \frac{\widetilde{\Theta} e}{k} \sin\left(\widetilde{\theta} - g\right) . \tag{31}$$

Solving equation (30) for e gives us:

$$e = \frac{k - \widetilde{r}}{\widetilde{r}\cos\left(\widetilde{\theta} - g\right)}, \tag{32}$$

and plugging that expression into equation (31) yields:

$$\begin{aligned}
\widetilde{R} &= \frac{\widetilde{\Theta}\left(k - \widetilde{r}\right)}{k\widetilde{r}} \tan\left(\widetilde{\theta} - g\right) \\
&= \frac{\widetilde{\Theta}\left(\widetilde{\Theta}^2/Mm^2 - \widetilde{r}\right)}{\widetilde{r}\widetilde{\Theta}^2/Mm^2} \tan\left(\widetilde{\theta} - g\right) \\
&= \frac{\widetilde{\Theta}^2 - Mm^2\widetilde{r}}{\widetilde{r}\widetilde{\Theta}} \tan\left(\widetilde{\theta} - g\right) . \qquad (33)
\end{aligned}$$

Therefore:

$$\tan\left(\widetilde{\theta} - g\right) = \frac{\widetilde{r}\widetilde{R}\widetilde{\Theta}}{\widetilde{\Theta}^2 - Mm^2\widetilde{r}},$$

$$g = \widetilde{\theta} - \tan^{-1}\frac{\widetilde{r}\widetilde{R}\widetilde{\Theta}}{\widetilde{\Theta}^2 - Mm^2\widetilde{r}}. \tag{34}$$

Using the above formula for $\tan\left(\widetilde{\theta} - g\right)$, we also get

$$\begin{aligned}\sin\left(\widetilde{\theta} - g\right) &= \frac{\tan\left(\widetilde{\theta} - g\right)}{\pm\sqrt{\tan^2\left(\widetilde{\theta} - g\right) + 1}} \\ &= \frac{\widetilde{r}\widetilde{R}\widetilde{\Theta}}{\pm\sqrt{\left(\widetilde{\Theta}^2 - Mm^2\widetilde{r}\right)^2 + \left(\widetilde{r}\widetilde{R}\widetilde{\Theta}\right)^2}},\end{aligned} \tag{35}$$

and consequently, since the eccentricity e is non-negative, we can conclude that

$$\begin{aligned}e &= \frac{\widetilde{R}\widetilde{\Theta}}{Mm^2\sin\left(\widetilde{\theta} - g\right)} \\ &= \frac{1}{Mm^2\widetilde{r}}\sqrt{\left(\widetilde{\Theta}^2 - Mm^2\widetilde{r}\right)^2 + \left(\widetilde{r}\widetilde{R}\widetilde{\Theta}\right)^2}.\end{aligned} \tag{36}$$

We now have formulas for both e and g in terms of the initial values of the dynamic variables. Thus, the orbital path, r in terms of θ, is fully described in terms of those initial values. Unfortunately, we cannot easily describe either r or θ in terms of time t. In fact, except when $e = 0$ (the circular case), both $r\,(t)$ and

$\theta(t)$ are transcendental functions. To deal with this problem, it is customary to introduce a form of *pseudo-time* $u(t)$, which is a monotone increasing function of time (known to astronomers as the *eccentric anomaly*). The function $u(t)$ is defined by an integral of the form

$$u(t) = c\int_T^t \frac{1}{r(\tau)}d\tau, \tag{37}$$

where the scaling constant c is chosen differently in the two cases, $e = 1$ (parabolic) and $e \neq 1$ (elliptic or hyperbolic); and T is generally taken to be the time of perihelion (so that $u = 0$ at perihelion, $t = T$). (See **[13]**, page 17.) In the elliptic and hyperbolic cases, we choose

$$c = \sqrt{\frac{M}{a}} = \frac{Mm}{\left|\widetilde{\Theta}\right|}\sqrt{|e^2-1|}, \tag{38}$$

where a is the length of the semi-major axis of the conic, which is given by

$$a = \frac{\widetilde{\Theta}^2}{Mm^2\left|e^2-1\right|}. \tag{39}$$

It then turns out that, while the function $u(t)$ is transcendental, the inverse function $t(u)$ is given explicitly in the hyperbolic case by

$$t(u) = T + \frac{e\sinh u - u}{n}, \tag{40}$$

where the constant n, called the *mean motion*, is defined to be

$$n = M^{1/2}a^{-3/2} = \frac{M^2m^3\left|e^2-1\right|^{3/2}}{\widetilde{\Theta}^3}. \tag{41}$$

(In the elliptic case, the constant n is in fact the mean angular velocity, and the period of the orbit is $2\pi/n$.)

Furthermore, in the hyperbolic case the radial distance r is an explicit function of u, given by

$$r = a\left(e\cosh u - 1\right)\,. \tag{42}$$

(See **[13]**, page 21.) Now let u_0 be the value of u at $t = 0$. Plugging u_0 into equation (40) for $t\left(u\right)$ gives us

$$T = \frac{u_0 - e\sinh u_0}{n}\,. \tag{43}$$

But we also have (see **[13]**, page 23):

$$\widetilde{r}\frac{\widetilde{R}}{m} = \sqrt{Mae}\sinh u_0, \tag{44}$$

$$\sinh u_0 = \frac{\widetilde{r}\widetilde{R}}{m\sqrt{Mae}} = \frac{\widetilde{r}\widetilde{R}}{\left|\widetilde{\Theta}\right| e}\sqrt{e^2-1}, \tag{45}$$

$$u_0 = \sinh^{-1}\left(\frac{\widetilde{r}\widetilde{R}}{\left|\widetilde{\Theta}\right|}\sqrt{\frac{e^2-1}{e^2}}\right)\,. \tag{46}$$

Inserting the above formulas for u_0 and $\sinh u_0$ (equations (46) and (45)) and the definition of n (equation (41)) into equation (43), for T in terms of u_0 and n, we get

$$
\begin{aligned}
T &= \frac{\widetilde{\Theta}^3}{M^2 m^3 \left(e^2-1\right)^{3/2}} \left(\sinh^{-1}\left(\frac{\widetilde{r}\widetilde{R}}{\left|\widetilde{\Theta}\right|} \sqrt{\frac{e^2-1}{e^2}} \right) - \frac{\widetilde{r}\widetilde{R}}{\left|\widetilde{\Theta}\right|} \sqrt{e^2-1} \right) \\
&= \frac{\widetilde{\Theta}^3 \widetilde{r}}{M^2 m^3 \left|\widetilde{\Theta}\right| \left(e^2-1\right)} \left(\frac{\left|\widetilde{\Theta}\right|}{\widetilde{r}\sqrt{e^2-1}} \sinh^{-1}\left(\frac{\widetilde{r}\widetilde{R}}{\left|\widetilde{\Theta}\right|} \sqrt{\frac{e^2-1}{e^2}} \right) - \widetilde{R} \right) \\
&= \frac{\left(\operatorname{sign}\widetilde{\Theta}\right) m\widetilde{r}^3}{\widetilde{\Theta}^2 - 2Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2} \left(\frac{Mm^2}{\sqrt{\widetilde{\Theta}^2 - 2Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2}} \sinh^{-1}\Phi - \widetilde{R} \right), \quad (47)
\end{aligned}
$$

where:

$$
\Phi = \frac{\widetilde{r}\widetilde{R}\sqrt{\widetilde{\Theta}^2 - 2Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2}}{\sqrt{\left(\widetilde{\Theta}^2 - Mm^2\widetilde{r}\right)^2 + \left(\widetilde{r}\widetilde{R}\widetilde{\Theta}\right)^2}}. \quad (48)
$$

Using the identity $\tanh x = \sinh x/\sqrt{1+\sinh^2 x}$ we can also write this result in somewhat more attractive form as

$$
T = \frac{\left(\operatorname{sign}\widetilde{\Theta}\right) m\widetilde{r}^3}{\widetilde{\Theta}^2 - 2Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2} \left(\frac{Mm^2}{\sqrt{\widetilde{\Theta}^2 - 2Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2}} \tanh^{-1}\Omega - \widetilde{R} \right). \quad (49)
$$

where:

$$
\Omega = \frac{\widetilde{r}\widetilde{R}\sqrt{\widetilde{\Theta}^2 - 2Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2}}{\widetilde{\Theta}^2 - Mm^2\widetilde{r} + \left(\widetilde{r}\widetilde{R}\right)^2}. \quad (50)
$$

We also need to have an equation for θ as a function of u. Using our two formulas for r, equation (27) as a function of θ and equation (42) as a function of u, we get

$$\frac{k}{r} = 1 + e\cos(\theta - g) = \frac{k}{a(e\cosh u - 1)} = \frac{e^2 - 1}{e\cosh u - 1}, \tag{51}$$

$$e\cos(\theta - g) = \frac{e^2 - 1}{e\cosh u - 1} - 1 = \frac{e^2 - e\cosh u}{e\cosh u - 1}, \tag{52}$$

$$\cos(\theta - g) = \frac{e - \cosh u}{e\cosh u - 1}. \tag{53}$$

$$\sin(\theta - g) = \pm\sqrt{1 - \cos^2(\theta - g)} = \operatorname{sign}\widetilde{\Theta}\frac{\sqrt{e^2 - 1}\sinh u}{e\cosh u - 1}. \tag{54}$$

Note that the factor $\operatorname{sign}\widetilde{\Theta}$ adjusts for the direction of motion (counter-clockwise when $\widetilde{\Theta}$ is positive and clockwise when $\widetilde{\Theta}$ is negative). Solving for θ yields

$$\theta = g + \operatorname{sign} u \operatorname{sign}\widetilde{\Theta}\cos^{-1}\left(\frac{e - \cosh u}{e\cosh u - 1}\right). \tag{55}$$

Here the factor $\operatorname{sign} u$ adjusts the principal value of $\cos^{-1}$ to allow for motion on both sides of the semi-major axis, and the factor $\operatorname{sign}\widetilde{\Theta}$ adjusts the principal value of $\cos^{-1}$ in accordance with the direction of motion. When the motion is in the first and fourth quadrants, we also have

$$\theta = g + \sin^{-1}\left(\operatorname{sign}\widetilde{\Theta}\frac{\sqrt{e^2 - 1}\sinh u}{e\cosh u - 1}\right). \tag{56}$$

(Add $\pi/2$ to the above when the motion is in the second quadrant, and subtract

$\pi/2$ from the above when the motion is in the third quadrant.)

Finally, note that the asymptotic angles of the hyperbolic trajectory, with respect to the conic's axis of symmetry, are related to e by pure geometry. They are:

$$f_{\pm\infty} = (\theta - g)_{\pm\infty} = \pm \cos^{-1}(-1/e) \ . \tag{57}$$

5. Monodromy Matrix, Linearized Kepler Operator

Definition: Suppose that we have a first-order ODE system of the form

$$\dot{\mathbf{x}} = f\left(t, \mathbf{x}\right),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\dot{\mathbf{x}} = d\mathbf{x}/dt$. Let $\mathbf{x} = \varphi\left(t, \widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)$ be the *flow* of the ODE. That is, φ is the unique general solution of the ODE, such that $\varphi\left(0, \widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right) = \left(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)$ (in other words, $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ are the initial values of $x_1, x_2, \ldots, x_n$ at $t = 0$). Then we define the *monodromy matrix* M of the ODE in the neighborhood of $t = 0$ to be the $n \times n$ Jacobian matrix of partial derivatives of φ with respect to $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$. (See **[10]**, page 21.)

Notation: Throughout this discussion, we will use the following abbreviated notations for this (and other) Jacobian matrices of partial derivatives:

$$M = \frac{\partial \varphi}{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)} = \frac{\partial\left(x_1, x_2, \ldots, x_n\right)}{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)}.$$

The monodromy matrix has several important properties. For one thing, it is the matrix of first-order coefficients in the Taylor expansion of the general solution φ as a function of the small changes in initial values $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$:

$$
\begin{aligned}
\varphi\left(t, \widetilde{x}_1+\delta \widetilde{x}_1, \widetilde{x}_2+\delta \widetilde{x}_2, \ldots, \widetilde{x}_n+\delta \widetilde{x}_n\right) &= \varphi\left(t, \widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right) \\
&\quad + \frac{\partial\left(x_1, x_2, \ldots, x_n\right)}{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)} \begin{bmatrix} \delta \widetilde{x}_1 \\ \delta \widetilde{x}_2 \\ \vdots \\ \delta \widetilde{x}_n \end{bmatrix} \\
&\quad + R_1\left(t, \delta \widetilde{x}_1, \delta \widetilde{x}_2, \ldots, \delta \widetilde{x}_n\right) \\
&= \widetilde{\varphi}(t) + M \begin{bmatrix} \delta \widetilde{x}_1 \\ \delta \widetilde{x}_2 \\ \vdots \\ \delta \widetilde{x}_n \end{bmatrix} + R_1, \qquad (58)
\end{aligned}
$$

where $\widetilde{\varphi}(t) = \varphi\left(t, \widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)$ and the remainder R_1 is second-order or smaller in the $\delta \widetilde{x}_i$ variables.

Thus, the monodromy matrix is useful in estimating perturbations of the general solution due to small changes in the initial value parameters.

Another important property of the monodromy matrix is obtained by plugging the general solution φ into the original ODE and differentiating both sides with respect to $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$. On the left-hand side, we interchange differentiation with respect to $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ with differentiation with respect to t; on the right-hand side, we apply the Chain Rule. The result is:

$$\dot{M}(t) = A(t)\, M(t), \qquad (59)$$

where:

$$A(t) = \frac{\partial f(t,\mathbf{x})}{\partial(x_1,x_2,\ldots,x_n)} \circ \varphi(t,\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)$$

$$= \left.\frac{\partial f(t,\mathbf{x})}{\partial(x_1,x_2,\ldots,x_n)}\right|_{\varphi(t,\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)}, \tag{60}$$

which says that each column of M is a solution to the *linearized* ODE system

$$\frac{d\mathbf{x}}{dt}(t) = A(t)\,\mathbf{x}(t) = A(t)\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}. \tag{61}$$

Furthermore, differentiating $\varphi(0,\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n) = (\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)$ gives us

$$M(0) = \frac{\partial\varphi(0,\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)}{\partial(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)} = \frac{\partial(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)}{\partial(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)} = I, \tag{62}$$

the identity matrix. This tells us that, in fact, the n columns of M are the n linearly independent solutions to the linearized ODE system and M is therefore the *fundamental matrix solution* to the linearized ODE system. (See [**10**], page 21.)

Thus, in principle, there are two ways to obtain the monodromy matrix. The direct method, if the general solution to the original non-linear ODE is available, is to differentiate the general solution with respect to its initial value parameters. The indirect method is to solve the related linearized ODE system. If it should happen that the matrix $A(t) = A$ is independent of time, then linear ODE theory tells us that

$$M = \exp(At), \tag{63}$$

and furthermore, the eigenvalues of M can be obtained by taking $\exp\lambda$, where λ is an eigenvalue of A (see [**7**]). Unfortunately, it is rarely the case that A is time-independent. (One special case in which A is in fact time-independent is the Kepler problem when $\mathbf{x}(t)$ is a circular solution. The matrix A is time-independent in this case because the Hamiltonian is independent of θ, and, for a circular solution, the remaining variables r, R, and Θ are all constant.)

Now let us consider the case in which the ODE is a Hamiltonian system, with a time-independent Hamiltonian $H(\mathbf{z})$, where $\mathbf{z} \in \mathbb{R}^{2n}$. As before, let J be the standard symplectic 2-form and J^* be the corresponding isomorphism from the dual space of $\mathbb{R}^{2n}$ onto $\mathbb{R}^{2n}$. The Hamiltonian differential equation is $\dot{\mathbf{z}} = J^*(dH(\mathbf{z}))$. If we let $\mathbf{z} = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, then in matrix form, this is

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_2 \\ \dot{y}_1 \\ \vdots \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} dH(x_1, \ldots, x_n, y_1, \ldots, y_n), \tag{64}$$

where $dH(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is written as a column vector (for the purpose of matrix multiplication). The linearized ODE system is then:

$$\frac{d\mathbf{z}}{dt} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \left. \frac{\partial \left(dH \left(x_1, \ldots, x_n, y_1, \ldots, y_n \right) \right)}{\partial \left(x_1, \ldots, x_n, y_1, \ldots, y_n \right)} \right|_{\varphi(t, \widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{y}_1, \ldots, \widetilde{y}_n)} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_2 \\ \vdots \\ y_n \end{bmatrix} . \tag{65}$$

Since the time derivative $d\mathbf{z}/dt$ is also a linear operator, which we denote by $D\mathbf{z}$, we can put all the linear operators on the left-hand side and write

$$\left(D - J^* A \right) \mathbf{z} = 0, \tag{66}$$

where

$$A = \left. \frac{\partial \left(dH \left(x_1, \ldots, x_n, y_1, \ldots, y_n \right) \right)}{\partial \left(x_1, \ldots, x_n, y_1, \ldots, y_n \right)} \right|_{\varphi(t, \widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{y}_1, \ldots, \widetilde{y}_n)} . \tag{67}$$

Thus, in this setting, the monodromy matrix consists of the $2n$ linearly independent functions that span the kernel of the linear operator $D - J^* A$.

We now return to the Kepler problem in the plane, in Cartesian coordinates. The Hamiltonian ODE in this case is the Cartesian form of Newton's equations of motion:

$$\dot{x}_1 = \frac{y_1}{m} \tag{68}$$

$$\dot{x}_2 = \frac{y_2}{m} \tag{69}$$

$$\dot{y}_1 = -\frac{Mmx_1}{\|\mathbf{x}\|^3} \tag{70}$$

$$\dot{y}_2 = -\frac{Mmx_2}{\|\mathbf{x}\|^3}\,. \tag{71}$$

In this case, since the conjugate momentum variables are defined rather simply as $y_1 = m\dot{x}_1$ and $y_2 = m\dot{x}_2$, it is convenient to eliminate the momentum variables and write the ODE system in the original Newtonian (non-Hamiltonian) formulation as a second-order ODE system:

$$\ddot{x}_1 = -\frac{Mx_1}{\|\mathbf{x}\|^3} \tag{72}$$

$$\ddot{x}_2 = -\frac{Mx_2}{\|\mathbf{x}\|^3}\,. \tag{73}$$

Then we can write the (non-linear) Kepler equation as:

$$D^2\mathbf{x} = -\frac{M}{\|\mathbf{x}\|^3}\mathbf{x}, \tag{74}$$

and the kernel of the *linearized Kepler operator* $D^2 - K$ is thus the set of $\mathbb{R}^2$-valued functions $\mathbf{x} \in C^2$ such that

$$\left(D^2 - K\right)\mathbf{x} = 0, \tag{75}$$

where the linear operator K is defined as multiplication by the following time-dependent matrix:

$$\begin{aligned} K(t) &= \left.\frac{\partial\left(-Mx_1/\|\mathbf{x}\|^3, -Mx_2/\|\mathbf{x}\|^3\right)}{\partial(x_1, x_2)}\right|_{(x_1,x_2)=\varphi(t,\widetilde{x}_1,\widetilde{x}_2,\widetilde{y}_1,\widetilde{y}_2)} \\ &= \frac{M}{\|\mathbf{x}\|^5}\begin{bmatrix} 2x_1^2 - x_2^2 & 3x_1x_2 \\ 3x_1x_2 & 2x_2^2 - x_1^2 \end{bmatrix}_{(x_1,x_2)=\varphi(t,\widetilde{x}_1,\widetilde{x}_2,\widetilde{y}_1,\widetilde{y}_2)} . \end{aligned} \tag{76}$$

Since the second-order system is equivalent to the first-order system that it replaces, (the first two rows of) the monodromy matrix still provides the four linearly independent solutions for the (second-order) linearized Kepler operator.

In particular, hyperbolic solutions to the Kepler problem are asymptotically linear in t at both $-\infty$ and $+\infty$. Thus, when the initial-value parameters $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{y}_1$, and $\widetilde{y}_2$ are chosen so that $\varphi(t, \widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)$ is hyperbolic, it follows that the uniform norm of K is asymptotically proportional to $|1/t^3|$ at both $-\infty$ and $+\infty$. We will show in Section 8 that this bound implies that K is a compact operator between certain pairs of function spaces (in particular, between the δ-weighted spaces C^2_δ and $C^0_{\delta-2}$, which we shall define in Section 8).

6. Solutions to the Linearized Kepler Operator

In Cartesian coordinates, for the Kepler problem in the plane, we have a two-dimensional position vector $\mathbf{x} = (x_1, x_2)$ and a two-dimensional conjugate momentum vector $\mathbf{y} = (y_1, y_2)$. The general solution to the Kepler problem, formulated as a Hamiltonian system, is a four-dimensional vector-valued function $\varphi(t, \widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2) = (\mathbf{x}, \mathbf{y})(t, \widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)$, where t is time and $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{y}_1$, and $\widetilde{y}_2$ are the initial values of the dynamic variables at $t = 0$. The monodromy matrix in these coordinates would be the Jacobian matrix of partial derivatives

$$\frac{\partial(x_1, x_2, y_1, y_2)}{\partial(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)}. \tag{77}$$

As explained previously, this matrix is the fundamental matrix solution to the linearized Kepler operator, and its four columns are the four linearly independent solutions to that linear ODE system. Now for this particular type of Hamiltonian, in Cartesian coordinates, the momentum vector is simply $\mathbf{y} = m\dot{\mathbf{x}}$; consequently, the second two rows of the above matrix can be obtained by simply differentiating the first two rows with respect to time. Thus, in our subsequent investigation we will calculate only the first two rows, which reduces our investigation to the matrix

$$\frac{\partial(x_1, x_2)}{\partial(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)}. \tag{78}$$

Omitting the last two rows, in going from (77) to (78), does not affect the linear independence of the (truncated) four columns (as functions). That follows because differentiation (with respect to time) is itself a linear operation, so if a linear dependency existed among the *truncated* four columns ($\mathbb{R}^2$-valued functions), it would automatically extend to a linear dependency of the *full* four columns ($\mathbb{R}^4$-valued functions), using the exact same scalar coefficients.

While the monodromy matrix always gives us a complete set of linearly independent solutions to the linearized operator, these solutions are not necessarily the ones that will facilitate an investigation of the properties of the solutions. Other linear combinations of the solutions are possible, and one way to obtain them is to change the set of "initial-value parameters" that define a "general solution", and with respect to which we differentiate that general solution. In other words, rather than using the standard initial-value parameters $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{y}_1$, and $\widetilde{y}_2$, some other set of parameters that uniquely define each possible solution may yield results that are more amenable to investigation. In particular, we would like to know if any of these solutions are bounded.

Proposition 6.1: Let $\mathbf{x}(t, T, e, g, \widetilde{\Theta})$ be the general (hyperbolic) solution to the Kepler problem, where perihelion occurs at $t = T$, $e > 1$ is the eccentricity, g is the angle of inclination of the semi-major axis, and $\widetilde{\Theta}$ is the angular momentum. (Here $\mathbf{x}$ is a vector in the plane.) The partial derivative $\partial\mathbf{x}/\partial T$, considered as an $\mathbb{R}^2$-valued functions of the first variable t, is a bounded solution to the linearized Kepler operator.

Proof: The fact that $\partial\mathbf{x}/\partial T$ is a solution to the linearized Kepler operator follows by the same argument are for the monodromy matrix (plug the solution into the Kepler equation (74), differentiate both sides with respect to T, and interchange the order of differentiation on the left-hand side).

We know that solutions are invariant under time translation, hence $\mathbf{x}(t-T, 0, e, g, \widetilde{\Theta})$ is also a solution and it follows by uniqueness that $\mathbf{x}(t, T, e, g, \widetilde{\Theta}) = \mathbf{x}(t - T, 0, e, g, \widetilde{\Theta})$. Therefore $\partial\mathbf{x}/\partial T = -d\mathbf{x}/dt = -\dot{\mathbf{x}}$. But we know that the velocity of the hyperbolic solution is bounded, hence so is $\partial\mathbf{x}/\partial T$.

■

Comment: The choice of the "parameter set" is important here. The above proof depends on the fact that when you replace t with $t - T$, specifically $\mathbf{x}(t - T, 0, e, g, \widetilde{\Theta})$, the resulting solution (which is always "a" solution by time invariance) is actually the *same* solution as $\mathbf{x}(t, T, e, g, \widetilde{\Theta})$ – not just "some other solution that has the same time of perihelion T". In general, one might have to adjust the other three parameters to get the same solution, *i.e.*

$$\mathbf{x}(t, T, e(0), g(0), \widetilde{\Theta}(0)) = \mathbf{x}(t - T, 0, e(T), g(T), \widetilde{\Theta}(T)), \tag{79}$$

... and that would invalidate the proof. But our choice of parameters (eccentricity, the angle of inclination of the semi-major axis, and the angular momentum) completely determine the solution (trajectory and velocity) in all respects other than the "time origin". These parameters are "constants of the motion", and are independent of the time origin. This makes the proof work. (It is possible to choose the other three parameters "badly", for instance by picking the initial values of three of the dynamic variables at time $t = 0$, as is commonly done.)

There are other good "parameter sets". One parameter should probably always be chosen to be g (traditional celestial mechanics symbol for the angle of inclination of the semi-major axis). The other two parameters can be chosen from among $\widetilde{H}$ (energy), a (the semi-major axis), v_∞ (speed at infinity), $\widetilde{\Theta}$ (angular momentum), e (eccentricity), and r_{min} (the distance from the focus at perihelion).

Since $R = m\dot{r} = m\,(dr/dt) = 0$ at r_{min}, the polar coordinate formula (18) for H gives an equation for H in terms of r_{min} and $\Theta = \widetilde{\Theta}$. The polar coordinate formula (26) for a hyperbola relates r to $\widetilde{\Theta}$, e, and f (f being the traditional celestial mechanics symbol for the "true anomaly" $\theta - g$). Setting $f = 0$ gives an equation for r_{min} in terms of $\widetilde{\Theta}$ and e. Eliminating r_{min} between these two equations gives $H = \widetilde{H}$ in terms of $\widetilde{\Theta}$ and e. The result is:

$$\widetilde{H} = \frac{M^2 m^3\,(e^2 - 1)}{2\widetilde{\Theta}^2} = \frac{Mm}{2a}\,. \tag{80}$$

Note that there is a simple relationship between $\widetilde{H}$ and a. One could also substitute the "terminal velocity" v_∞ (speed at infinity) for $\widetilde{H}$.

Since there is a one-to-one relation between e and the angle of opening of the hyperbola ($2\cos^{-1}(-1/e)$), one can substitute that angle for e (although there is no particular advantage in doing so, other than the fact that the angle is more directly "geometric").

For our investigation, we shall choose the parameter set to be T, g, e, and $\widetilde{\Theta}$. So we will actually calculate the matrix of partial derivatives

$$\frac{\partial\,(x_1, x_2)}{\partial\left(T, e, g, \widetilde{\Theta}\right)}\,.$$

This matrix has two rows and four columns.

Note: Our chosen parameters T, g, e, and $\widetilde{\Theta}$ are actually very close to the classical Delaunay variables g, G, ℓ, and L. The correspondence is $g = g$, $G = \widetilde{\Theta}$, $\ell = 2\pi/a^{3/2}T$, and $L = a^2$. In 1808, Lagrange showed that (in modern language) the Delaunay variables are symplectic. (See [**1**].)

Proposition 6.2: Let $\mathbf{x}(t, T, e, g, \widetilde{\Theta})$ be the general (hyperbolic) solution to the Kepler problem, where perihelion occurs at $t = T$, $e > 1$ is the eccentricity, g is the angle of inclination of the semi-major axis, and $\widetilde{\Theta}$ is the angular momentum. Each of the four columns of the matrix

$$\frac{\partial\left(x_1, x_2\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)}$$

is a solution to the linearized Kepler operator, and the columns are linearly independent (as $\mathbb{R}^2$-valued functions of the first variable t); hence they span the four-dimensional kernel of the linearized Kepler operator.

Proof: The fact that each column ($\partial\mathbf{x}/\partial T$, $\partial\mathbf{x}/\partial e$, $\partial\mathbf{x}/\partial g$, and $\partial\mathbf{x}/\partial\widetilde{\Theta}$) is a solution to the linearized Kepler operator follows by the same argument are for the monodromy matrix (plug the solution into the Kepler equation (74), differentiate both sides with respect to T, and interchange the order of differentiation on the left-hand side).

We know that the four columns of the (truncated) monodromy matrix

$$\frac{\partial\left(x_1, x_2\right)}{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2\right)}$$

are a complete set of linearly independent solutions to the linearized Kepler operator. But by the Chain Rule, we have

$$\frac{\partial\left(x_1, x_2\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)} = \frac{\partial\left(x_1, x_2\right)}{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2\right)} \frac{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)} . \tag{81}$$

So we need to show that the "change of variables" from $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{y}_1$, and $\widetilde{y}_2$ to T, g, e, and $\widetilde{\Theta}$ is non-singular. By the Chain Rule again, we know that

$$\frac{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)} = \frac{\partial\left(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2\right)}{\partial\left(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta}\right)} \frac{\partial\left(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta}\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)}, \tag{82}$$

where $\widetilde{r}$, $\widetilde{\theta}$, $\widetilde{R}$, and $\widetilde{\Theta}$ are the initial values of the dynamic position and momentum variables in polar coordinates. The initial values in the two coordinate systems, Cartesian and polar, are related by

$$\widetilde{x}_1 = \widetilde{r}\cos\widetilde{\theta}, \tag{83}$$

$$\widetilde{x}_2 = \widetilde{r}\sin\widetilde{\theta}, \tag{84}$$

$$\widetilde{y}_1 = \widetilde{R}\cos\widetilde{\theta} - \frac{\widetilde{\Theta}}{\widetilde{r}}\sin\widetilde{\theta}, \tag{85}$$

$$\widetilde{y}_2 = \widetilde{R}\sin\widetilde{\theta} + \frac{\widetilde{\Theta}}{\widetilde{r}}\cos\widetilde{\theta} . \tag{86}$$

Hence:

$$\frac{\partial(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)}{\partial(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta})} = \begin{bmatrix} \cos\widetilde{\theta} & -\widetilde{r}\sin\widetilde{\theta} & 0 & 0 \\ \sin\widetilde{\theta} & \widetilde{r}\cos\widetilde{\theta} & 0 & 0 \\ \dfrac{\widetilde{\Theta}}{\widetilde{r}^2}\sin\widetilde{\theta} & -\widetilde{R}\sin\widetilde{\theta} - \dfrac{\widetilde{\Theta}}{\widetilde{r}}\cos\widetilde{\theta} & \cos\widetilde{\theta} & -\dfrac{1}{\widetilde{r}}\sin\widetilde{\theta} \\ -\dfrac{\widetilde{\Theta}}{\widetilde{r}^2}\cos\widetilde{\theta} & \widetilde{R}\cos\widetilde{\theta} - \dfrac{\widetilde{\Theta}}{\widetilde{r}}\sin\widetilde{\theta} & \sin\widetilde{\theta} & \dfrac{1}{\widetilde{r}}\cos\widetilde{\theta} \end{bmatrix}, \tag{87}$$

$$\det\frac{\partial(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)}{\partial(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta})} = \widetilde{r}\left(\sin^2\widetilde{\theta} + \cos^2\widetilde{\theta}\right)\frac{1}{\widetilde{r}}\left(\sin^2\widetilde{\theta} + \cos^2\widetilde{\theta}\right) = 1, \tag{88}$$

provided, of course, that $\widetilde{r} \neq 0$ (this condition is met by hyperbolic solutions, and, in fact, by all solutions with $\widetilde{\Theta} \neq 0$).

Now, we have previously derived formulas for T, e, and g in terms of $\widetilde{r}$, $\widetilde{\theta}$, $\widetilde{R}$, and $\widetilde{\Theta}$. So, rather than directly calculate the transformation Jacobian matrix

$$\frac{\partial\left(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta}\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)},$$

we will calculate the Jacobian matrix A of the inverse transformation:

$$A = \frac{\partial\left(T, e, g, \widetilde{\Theta}\right)}{\partial\left(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta}\right)} = \left[\frac{\partial\left(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta}\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)}\right]^{-1}. \tag{89}$$

Recall that T, e, and $\widetilde{\Theta}$ do not depend on $\widetilde{\theta}$; and g depends on $\widetilde{\theta}$ in the following simple manner:

$$g = \widetilde{\theta} - \tan^{-1} \frac{\widetilde{r}\widetilde{R}\widetilde{\Theta}}{\widetilde{\Theta}^2 - Mm^2\widetilde{r}}. \tag{90}$$

Furthermore, $\widetilde{\Theta}$ depends only on itself, namely $\widetilde{\Theta} = \widetilde{\Theta}$. These two facts give us the second column and bottom row of A, respectively, so we can write A as follows:

$$A = \frac{\partial\left(T, e, g, \widetilde{\Theta}\right)}{\partial\left(\widetilde{r}, \widetilde{\theta}, \widetilde{R}, \widetilde{\Theta}\right)} = \begin{bmatrix} \frac{\partial T}{\partial \widetilde{r}} & 0 & \frac{\partial T}{\partial \widetilde{R}} & \mathrm{X} \\ \frac{\partial e}{\partial \widetilde{r}} & 0 & \frac{\partial e}{\partial \widetilde{R}} & \mathrm{X} \\ \mathrm{X} & 1 & \mathrm{X} & \mathrm{X} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{91}$$

where the “X” entries play no role in the calculation of the determinant. Hence

$$\det A = \frac{\partial T}{\partial \widetilde{R}}\frac{\partial e}{\partial \widetilde{r}} - \frac{\partial T}{\partial \widetilde{r}}\frac{\partial e}{\partial \widetilde{R}}. \tag{92}$$

Using our formulas for T and e in terms of the initial value parameters $\widetilde{r}$, $\widetilde{\theta}$, $\widetilde{R}$, and $\widetilde{\Theta}$, we calculate:

$$\begin{aligned}\frac{\partial T}{\partial \widetilde{r}} &= -\left(\left(\widetilde{R}\sqrt{\widetilde{\Theta}^2 - 2m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2}\left(-5m^6M^3\widetilde{r}^3 + \widetilde{r}^4\widetilde{R}^4\widetilde{\Theta}^2 + 4\widetilde{r}^2\widetilde{R}^2\widetilde{\Theta}^4\right.\right.\right.\\
&\quad \left. +3\widetilde{\Theta}^6 - m^2M\widetilde{r}\widetilde{\Theta}^2\left(7\widetilde{r}^2\widetilde{R}^2 + 11\widetilde{\Theta}^2\right) + m^4M^2\widetilde{r}^2\left(\widetilde{r}^2\widetilde{R}^2 + 14\widetilde{\Theta}^2\right)\right)\\
&\quad +3m^2M\left(m^2M\widetilde{r} - \widetilde{\Theta}^2\right)\left(m^4M^2\widetilde{r}^2 - 2m^2M\widetilde{r}\widetilde{\Theta}^2 + \widetilde{r}^2\widetilde{R}^2\widetilde{\Theta}^2 + \widetilde{\Theta}^4\right)\\
&\quad \left.\left.\tanh^{-1}\frac{\widetilde{r}\widetilde{R}\sqrt{\widetilde{\Theta}^2 - 2m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2}}{\widetilde{\Theta}^2 - m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2}\right) m\widetilde{r}^2 \operatorname{sign}\widetilde{\Theta}\right)\Big/\\
&\quad \left(\left(\widetilde{\Theta}^2 - 2m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2\right)^{5/2}\left(m^4M^2\widetilde{r}^2 - 2m^2M\widetilde{r}\widetilde{\Theta}^2 + \widetilde{r}^2\widetilde{R}^2\widetilde{\Theta}^2 + \widetilde{\Theta}^4\right)\right),
\end{aligned} \tag{93}$$

$$\begin{aligned}\frac{\partial T}{\partial \widetilde{R}} &= \left(m\widetilde{r}^3 \operatorname{sign}\widetilde{\Theta}\left(\sqrt{\widetilde{\Theta}^2 - 2m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2}\left(4m^6M^3\widetilde{r}^3 + \widetilde{r}^4\widetilde{R}^4\widetilde{\Theta}^2 - \widetilde{\Theta}^6\right.\right.\right.\\
&\quad \left. +m^4M^2\widetilde{r}^2\left(\widetilde{r}^2\widetilde{R}^2 - 8\widetilde{\Theta}^2\right) + m^2M\widetilde{r}\widetilde{\Theta}^2\left(\widetilde{r}^2\widetilde{R}^2 + 5\widetilde{\Theta}^2\right)\right)\\
&\quad -3m^2M\widetilde{r}^2\widetilde{R}\left(m^4M^2\widetilde{r}^2 - 2m^2M\widetilde{r}\widetilde{\Theta}^2 + \widetilde{r}^2\widetilde{R}^2\widetilde{\Theta}^2 + \widetilde{\Theta}^4\right)\\
&\quad \left.\left.\left.\tanh^{-1}\frac{\widetilde{r}\widetilde{R}\sqrt{\widetilde{\Theta}^2 - 2m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2}}{\widetilde{\Theta}^2 - m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2}\right)\right)\right)\Big/\\
&\quad \left(\left(\widetilde{\Theta}^2 - 2m^2M\widetilde{r} + \widetilde{r}^2\widetilde{R}^2\right)^{5/2}\left(m^4M^2\widetilde{r}^2 - 2m^2M\widetilde{r}\widetilde{\Theta}^2 + \widetilde{r}^2\widetilde{R}^2\widetilde{\Theta}^2 + \widetilde{\Theta}^4\right)\right),
\end{aligned} \tag{94}$$

$$\frac{\partial e}{\partial \widetilde{r}} = \frac{m^2M\widetilde{r}\widetilde{\Theta}^2 - \widetilde{\Theta}^4}{m^2M\widetilde{r}^2\sqrt{\widetilde{r}^2\widetilde{R}^2\widetilde{\Theta}^2 + \left(\widetilde{\Theta}^2 - m^2M\widetilde{r}\right)^2}}, \tag{95}$$

$$\frac{\partial e}{\partial \widetilde{R}} = \frac{\widetilde{r}\widetilde{R}\widetilde{\Theta}^2}{m^2 M \sqrt{\widetilde{r}^2 \widetilde{R}^2 \widetilde{\Theta}^2 + \left(\widetilde{\Theta}^2 - m^2 M \widetilde{r}\right)^2}} \,. \tag{96}$$

Multiplying these values in pairs and simplifying, we get the following rather remarkable result:

$$\det A = \frac{\widetilde{r}\widetilde{\Theta}^2 \operatorname{sign} \widetilde{\Theta}}{m M \sqrt{\widetilde{r}^2 \widetilde{R}^2 \widetilde{\Theta}^2 + \left(\widetilde{\Theta}^2 - m^2 M \widetilde{r}\right)^2}} = \frac{\widetilde{\Theta}^2 \operatorname{sign} \widetilde{\Theta}}{m^3 M^2 e} \,. \tag{97}$$

Note that for the hyperbolic case ($\widetilde{\Theta} \neq 0$ and $e > 1$), this determinant indicates that the transformation (change of variables) is non-singular. (For the straight-line case, $\widetilde{\Theta} = 0$ and the above transformation *is* singular. This should not be a surprise, because in the straight-line case $e = 1$, so the remaining three parameters T, e, and g are *not* a complete set of parameters. One would have to choose some other parameter, such as energy H, to replace e in the straight-line case.)

Therefore, we have shown that the four columns of the 2×4 matrix of partial derivatives

$$\frac{\partial \left(x_1, x_2\right)}{\partial \left(T, e, g, \widetilde{\Theta}\right)}$$

gives us the complete set of four linearly independent solutions of the linearized Kepler operator.

■

7. Actual Solutions and Their Properties

Recall that the constant a, the length of the semi-major axis of the conic, is given by

$$a = \frac{\widetilde{\Theta}^2}{Mm^2 \left| e^2 - 1 \right|} \, . \tag{98}$$

In the hyperbolic case, $e^2 > 1$, so we can remove the absolute value bars in the above. Now, although a is a constant for any particular solution, it can also be considered as a function of the parameters e and $\widetilde{\Theta}$, so we can write:

$$a \left(e, \widetilde{\Theta} \right) = \frac{\widetilde{\Theta}^2}{Mm^2 \left(e^2 - 1 \right)} \, . \tag{99}$$

Hence

$$\frac{\partial a}{\partial e} = - \frac{2e\widetilde{\Theta}^2}{\left(e^2 - 1 \right)^2 m^2 M} , \tag{100}$$

and

$$\frac{\partial a}{\partial \widetilde{\Theta}} = \frac{2\widetilde{\Theta}}{\left(e^2 - 1 \right) m^2 M} \, . \tag{101}$$

Recall that the constant n, called the *mean motion*, is defined to be

$$n = M^{1/2} a^{-3/2} = \frac{M^2 m^3 \left| e^2 - 1 \right|^{3/2}}{\widetilde{\Theta}^3} \, . \tag{102}$$

In the hyperbolic case, considering n as a function of the parameters e and $\widetilde{\Theta}$, we can write:

$$n\left(e,\widetilde{\Theta}\right)=\frac{m^3M^2\left(e^2-1\right)^{3/2}}{\widetilde{\Theta}^3}\,. \tag{103}$$

Hence

$$\frac{\partial n}{\partial e}=\frac{3e\sqrt{e^2-1}m^3M^2}{\widetilde{\Theta}^3}, \tag{104}$$

and

$$\frac{\partial n}{\partial\widetilde{\Theta}}=-\frac{3\left(e^2-1\right)^{3/2}m^3M^2}{\widetilde{\Theta}^4}\,. \tag{105}$$

Recall that $t(u)$ is given explicitly in the hyperbolic case by

$$t(u)=T+\frac{e\sinh u-u}{n}\,. \tag{106}$$

Now let $t_p=t-T$ be the time measured from T, the time of perihelion. Then:

$$t_p\left(e,\widetilde{\Theta},u\right)=t-T=\frac{e\sinh u-u}{n\left(e,\widetilde{\Theta}\right)}=\frac{\widetilde{\Theta}^3(e\sinh u-u)}{\left(e^2-1\right)^{3/2}m^3M^2} \tag{107}$$

Another way that we can use the formula for $t(u)$ is to write a function F that implicitly defines u as a function of t, T, e, and $\widetilde{\Theta}$:

$$F\left(t,T,e,\widetilde{\Theta},u\right)=n\left(e,\widetilde{\Theta}\right)(t-T)+u-e\sinh u=0\,. \tag{108}$$

Then we have

$$\frac{\partial F}{\partial u}=1-e\cosh u<0, \tag{109}$$

where the inequality holds because $e > 1$ and $\cosh u \geq 1$. Therefore, the Implicit Function Theorem applies and we can use it to calculate the derivatives of u with respect to t, T, e, and $\widetilde{\Theta}$. Those calculations are as follows:

$$\frac{\partial F}{\partial t} = n\left(e, \widetilde{\Theta}\right), \tag{110}$$

$$\frac{\partial u}{\partial t} = -\frac{\frac{\partial F}{\partial t}}{\frac{\partial F}{\partial u}} = \frac{\left(e^2-1\right)^{3/2} m^3 M^2}{\widetilde{\Theta}^3 (e \cosh u - 1)}. \tag{111}$$

$$\frac{\partial F}{\partial T} = -n\left(e, \widetilde{\Theta}\right), \tag{112}$$

$$\frac{\partial u}{\partial T} = -\frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial u}} = \frac{\left(e^2-1\right)^{3/2} m^3 M^2}{\widetilde{\Theta}^3 (1 - e \cosh u)}. \tag{113}$$

$$\frac{\partial F}{\partial e} = (t-T)\frac{\partial n}{\partial e} - \sinh u = \frac{3e\sqrt{e^2-1}m^3 M^2 (t-T)}{\widetilde{\Theta}^3} - \sinh u, \tag{114}$$

$$\frac{\partial u}{\partial e} = -\frac{\frac{\partial F}{\partial e}}{\frac{\partial F}{\partial u}} = \frac{\frac{3e\sqrt{e^2-1}m^3 M^2 (t-T)}{\widetilde{\Theta}^3} - \sinh u}{e \cosh u - 1}. \tag{115}$$

$$\frac{\partial F}{\partial \widetilde{\Theta}} = (t-T)\frac{\partial n}{\partial \widetilde{\Theta}} = -\frac{3\left(e^2-1\right)^{3/2} m^3 M^2 (t-T)}{\widetilde{\Theta}^4}, \tag{116}$$

$$\frac{\partial u}{\partial \widetilde{\Theta}} = -\frac{\dfrac{\partial F}{\partial \widetilde{\Theta}}}{\dfrac{\partial F}{\partial u}} = \frac{3\left(e^2-1\right)^{3/2} m^3 M^2 (t-T)}{\widetilde{\Theta}^4 (1-e\cosh u)}\,. \tag{117}$$

Finally, note that as a function of the pseudo-time u and the parameters e and $\widetilde{\Theta}$, we have

$$r\left(e,\widetilde{\Theta},u\right) = a\left(e,\widetilde{\Theta}\right)(e\cosh u - 1) = \frac{\widetilde{\Theta}^2 (e\cosh u - 1)}{\left(e^2-1\right) m^2 M}, \tag{118}$$

and we know that $\theta\left(e,g,\widetilde{\Theta},u\right)$ satisfies

$$\cos\left(\theta - g\right) = \frac{e - \cosh u}{e\cosh u - 1}, \tag{119}$$

and

$$\sin\left(\theta - g\right) = \operatorname{sign}\widetilde{\Theta}\,\frac{\sqrt{e^2-1}\,\sinh u}{e\cosh u - 1}\,. \tag{120}$$

7.1 Calculation of the Derivatives of $\mathbf{x} = (x_{1g}, x_{2g})$

Using the preceding formula for $r\left(e,\widetilde{\Theta},u\right)$, together with the formulas for $\cos\left(\theta - g\right)$ and $\sin\left(\theta - g\right)$, we can easily write down equations for the Cartesian coordinates x_{1g} and x_{2g} of the body, where $\mathbf{x} = (x_{1g}, x_{2g})$. Here we use the subscripts $1g$ and $2g$ to indicate that these coordinates take into account the angle of inclination g:

$$
\begin{aligned}
x_{1g}\left(e,g,\widetilde{\Theta},u\right) &= r\left(e,\widetilde{\Theta},u\right)\cos\theta = r\left(e,\widetilde{\Theta},u\right)\cos\left(\theta-g+g\right) \\
&= r\left(e,\widetilde{\Theta},u\right)\left(\cos\left(\theta-g\right)\cos g-\sin\left(\theta-g\right)\sin g\right) \\
&= r\left(e,\widetilde{\Theta},u\right)\left(\frac{e-\cosh u}{e\cosh u-1}\cos g-\operatorname{sign}\widetilde{\Theta}\frac{\sqrt{e^2-1}\sinh u}{e\cosh u-1}\sin g\right),
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
x_{2g}\left(e,g,\widetilde{\Theta},u\right) &= r\left(e,\widetilde{\Theta},u\right)\sin\theta = r\left(e,\widetilde{\Theta},u\right)\sin\left(\theta-g+g\right) \\
&= r\left(e,\widetilde{\Theta},u\right)\left(\cos\left(\theta-g\right)\sin g+\sin\left(\theta-g\right)\cos g\right) \\
&= r\left(e,\widetilde{\Theta},u\right)\left(\frac{e-\cosh u}{e\cosh u-1}\sin g+\operatorname{sign}\widetilde{\Theta}\frac{\sqrt{e^2-1}\sinh u}{e\cosh u-1}\cos g\right).
\end{aligned}
\tag{122}
$$

Note that we can write the above as:

$$
\begin{bmatrix} x_{1g}\left(e,g,\widetilde{\Theta},u\right) \\ \\ x_{2g}\left(e,g,\widetilde{\Theta},u\right) \end{bmatrix} = \begin{bmatrix} \cos g & -\sin g \\ \\ \sin g & \cos g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 0 & \operatorname{sign}\widetilde{\Theta} \end{bmatrix} \begin{bmatrix} x_1\left(e,\widetilde{\Theta},u\right) \\ \\ x_2\left(e,\widetilde{\Theta},u\right) \end{bmatrix},
\tag{123}
$$

where u is an implicitly-defined function of t, T, e, and $\widetilde{\Theta}$, and:

$$\begin{bmatrix} x_1\left(e,\widetilde{\Theta},u\right) \\ \\ x_2\left(e,\widetilde{\Theta},u\right) \end{bmatrix} = r\left(e,\widetilde{\Theta},u\right)\begin{bmatrix} \dfrac{e-\cosh u}{e\cosh u-1} \\ \\ \dfrac{\sqrt{e^2-1}\sinh u}{e\cosh u-1} \end{bmatrix} = \begin{bmatrix} \dfrac{\widetilde{\Theta}^2(e-\cosh u)}{\left(e^2-1\right)m^2M} \\ \\ \dfrac{\widetilde{\Theta}^2\sinh u}{\sqrt{e^2-1}m^2M} \end{bmatrix}. \tag{124}$$

As expected, due to the rotational invariance of solutions to Newtonian gravitational problems, the angle of inclination g appears only in a "pure rotational" matrix of the form

$$\Psi(g) = \begin{bmatrix} \cos g & -\sin g \\ \\ \sin g & \cos g \end{bmatrix}. \tag{125}$$

Consequently, in the matrix formulas above, we have introduced the "pre-rotational" Cartesian coordinates (x_1, x_2).

The other simple linear transformation that "factors out" is

$$\begin{bmatrix} 1 & 0 \\ \\ 0 & \operatorname{sign}\widetilde{\Theta} \end{bmatrix}, \tag{126}$$

which is either the identity matrix or a reflection of the (pre-rotational) x_2 axis about the x_1 axis. This transformation accounts for the fact that the body could be moving either counter-clockwise ($\operatorname{sign}\widetilde{\Theta} = 1$) or clockwise ($\operatorname{sign}\widetilde{\Theta} = -1$)

along the hyperbola (this relates to the "time invariance" of solutions, the fact that solutions are invariant under the transformation $t \mapsto -t$).

Now, we want to calculate the derivatives of the solution (x_{1g}, x_{2g}) with respect to each of the four parameters T, e, g, and $\widetilde{\Theta}$. But g appears *only* in the factor matrix $\Psi(g)$, so, in the cases of T, e, and $\widetilde{\Theta}$ we can simplify the calculations by dealing with the pre-rotational coordinates (x_1, x_2). On the other hand, in the case of differentiation with respect to g, we need to only differentiate $\Psi(g)$:

$$\begin{aligned}
\frac{d\Psi(g)}{dg} &= \begin{bmatrix} -\sin g & -\cos g \\ \\ \cos g & -\sin g \end{bmatrix} \\
&= \begin{bmatrix} \cos\left(g + \frac{\pi}{2}\right) & -\sin\left(g + \frac{\pi}{2}\right) \\ \\ \sin\left(g + \frac{\pi}{2}\right) & \cos\left(g + \frac{\pi}{2}\right) \end{bmatrix} \\
&= \Psi\left(g + \frac{\pi}{2}\right) \\
&= \Psi(g)\,\Psi\left(\frac{\pi}{2}\right) \\
&= \Psi(g) \begin{bmatrix} 0 & -1 \\ \\ 1 & 0 \end{bmatrix}.
\end{aligned} \tag{127}$$

Therefore, the rotational matrix $\Psi(g)$ and the reflection matrix are common factors in all four cases. In particular, the derivatives with respect to g are calculated as follows:

$$\begin{bmatrix} \dfrac{dx_{1g}}{dg} \\ \\ \dfrac{dx_{2g}}{dg} \end{bmatrix} = \begin{bmatrix} \cos g & -\sin g \\ \\ \sin g & \cos g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 0 & \operatorname{sign} \widetilde{\Theta} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} \cos g & -\sin g \\ \\ \sin g & \cos g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 0 & \operatorname{sign} \widetilde{\Theta} \end{bmatrix} \begin{bmatrix} -x_2 \\ \\ x_1 \end{bmatrix} . \tag{128}$$

7.2 Calculation of the Derivatives of x_1

The derivatives of x_1 with respect to the parameters T, e, and $\widetilde{\Theta}$ can all be calculated by means of the Chain Rule, using the following derivative of x_1 with respect to pseudo-time u:

$$\frac{\partial x_1}{\partial u}\left(e, \widetilde{\Theta}, u\right) = -\frac{\widetilde{\Theta}^2 \sinh u}{(e^2-1)\, m^2 M} . \tag{129}$$

Recall that u is an implicitly-defined function of t, T, e, and $\widetilde{\Theta}$. We will make the customary "abuse of notation" and write x_1 for both the function $x_1\left(t, T, e, \widetilde{\Theta}\right)$ and the function $x_1\left(e, \widetilde{\Theta}, u\right)$ (hopefully, our meaning will be clear from context). Thus, the derivative of $x_1\left(t, T, e, \widetilde{\Theta}\right)$ with respect to T is:

$$\frac{\partial x_1}{\partial T}\left(t, T, e, \widetilde{\Theta}\right) = \frac{\partial x_1}{\partial u}\frac{\partial u}{\partial T} = \frac{mM\sqrt{e^2-1}\sinh u}{\widetilde{\Theta}(e\cosh u - 1)} . \tag{130}$$

We claim that $\partial x_1/\partial T$ is asymptotic to a constant at $t = +\infty$. The limit is easier to calculate if we make the substitution $u = \log w$. We know that $u \to +\infty$ as

$t \to +\infty$ (since u is a monotone increasing function of t, and in particular, u is asymptotic to $\log t$). It follows that $w = \exp u \to +\infty$ as $t \to +\infty$ as well (and in fact, w is asymptotic to t). Hence:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\partial x_1}{\partial T} &= \lim_{w\to+\infty} \frac{mM\sqrt{e^2-1}\sinh(\log w)}{\widetilde{\Theta}(e\cosh(\log w)-1)} \\
&= \lim_{w\to+\infty} \frac{\sqrt{e^2-1}mM\,(w^2-1)}{(ew^2-2w+e)\,\widetilde{\Theta}} \\
&= \frac{\sqrt{e^2-1}mM}{e\widetilde{\Theta}}\,. \qquad (131)
\end{aligned}$$

Note that at $t = T$ we have $u = 0$. Therefore:

$$\left.\frac{\partial x_1}{\partial T}\right|_{t=T} = 0\,. \qquad (132)$$

We claim that $\partial x_1/\partial T$ is asymptotic to a constant at $t = -\infty$. The limit is easier to calculate if we make the substitution $u = -\log(-w)$. We know that $u \to -\infty$ as $t \to -\infty$ (since u is a monotone increasing function of t, and in particular, u is asymptotic to $-\log(-t)$). It follows that $w = -\exp(-u) \to -\infty$ as $t \to -\infty$ as well (and in fact, w is asymptotic to t). Hence:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\partial x_1}{\partial T} &= \lim_{w\to+\infty} \frac{mM\sqrt{e^2-1}\sinh(-\log(-w))}{\widetilde{\Theta}(e\cosh(-\log(-w))-1)} \\
&= \lim_{w\to+\infty} -\frac{\sqrt{e^2-1}mM\,(w^2-1)}{(ew^2+2w+e)\,\widetilde{\Theta}} \\
&= -\frac{\sqrt{e^2-1}mM}{e\widetilde{\Theta}}\,. \qquad (133)
\end{aligned}$$

The derivative of $x_1\left(e,\widetilde{\Theta},u\right)$ with respect to e is:

$$\frac{\partial x_1}{\partial e}\left(e,\widetilde{\Theta},u\right)=-\frac{\widetilde{\Theta}^2\left(e^2-2e\cosh u+1\right)}{\left(e^2-1\right)^2 m^2 M}\,. \tag{134}$$

Thus, the derivative of $x_1\left(t,T,e,\widetilde{\Theta}\right)$ with respect to e is:

$$\begin{aligned}\frac{\partial x_1}{\partial e}\left(t,T,e,\widetilde{\Theta}\right) &= \frac{\partial x_1}{\partial e}\left(e,\widetilde{\Theta},u\right)+\frac{\partial x_1}{\partial u}\frac{\partial u}{\partial e}\\ &= \frac{\widetilde{\Theta}^2}{\left(e^2-1\right)^2 m^2 M}\Bigg(-e^2+2e\cosh u-1\\ &\quad+\frac{(e^2-1)\sinh u\left(3e\sqrt{e^2-1}m^3M^2(-t+T)+\widetilde{\Theta}^3\sinh u\right)}{\widetilde{\Theta}^3(e\cosh u-1)}\Bigg)\,.\end{aligned} \tag{135}$$

We claim that $\partial x_1/\partial e$ is asymptotically proportional to t at $t=+\infty$. More precisely, the ratio $\left(\partial x_1/\partial e\right)/t$ has a constant limit. For the denominator in this ratio, we use equation (107) for $t_p\left(e,\widetilde{\Theta},u\right)$, the time from perihelion. As usual, the limit is easier to calculate if we make the substitution $u=\log w$.

$$\begin{aligned}\lim_{t\to+\infty}\frac{\frac{\partial x_1}{\partial e}}{t_p} &= \lim_{w\to+\infty}\frac{2mMw}{\sqrt{e^2-1}\widetilde{\Theta}\left(ew^2-2w\log w-e\right)}\Bigg(-1-e^2+e\left(\frac{1}{w}+w\right)\\ &\quad-\frac{\left(e^2-1\right)\left(w^2-1\right)\left(6e\sqrt{e^2-1}m^3M^2(t-T)w-\left(w^2-1\right)\widetilde{\Theta}^3\right)}{2w\left(ew^2-2w+e\right)\widetilde{\Theta}^3}\Bigg)\\ &= \frac{\left(3e^2-1\right)mM}{e^2\sqrt{e^2-1}\widetilde{\Theta}}\,.\end{aligned} \tag{136}$$

At $t = T$ we have $u = 0$. Therefore:

$$\left.\frac{\partial x_1}{\partial e}\right|_{t=T} = -\frac{\widetilde{\Theta}^2}{(e+1)^2 m^2 M} \,. \tag{137}$$

Furthermore, $\partial x_1/\partial e$ is asymptotically proportional to t at $t = -\infty$:

$$\begin{aligned}
\lim_{t\to-\infty} \frac{\frac{\partial x_1}{\partial e}}{t_p} &= \lim_{w\to-\infty} \frac{2mMw}{\sqrt{e^2-1}\widetilde{\Theta}\left(ew^2+2w\log w-e\right)} \left(-1-e^2-\frac{e\left(w^2+1\right)}{w}\right. \\
&\quad \left. + \frac{\left(e^2-1\right)\left(w^2-1\right)\left(6e\sqrt{e^2-1}m^3M^2(t-T)w-\left(w^2-1\right)\widetilde{\Theta}^3\right)}{2w\left(ew^2+2w+e\right)\widetilde{\Theta}^3}\right) \\
&= \frac{\left(1-3e^2\right)mM}{e^2\sqrt{e^2-1}\widetilde{\Theta}} \,.
\end{aligned} \tag{138}$$

Now, the derivative $\partial x_{1g}/\partial g$ is effectively $-x_2$, as explained earlier. This, too, is asymptotically proportional to t at $t = +\infty$:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\frac{\partial x_{1g}}{\partial g}}{t_p} &= \lim_{t\to+\infty} \frac{-x_2}{t_p} \\
&= \lim_{w\to+\infty} -\frac{\left(e^2-1\right)mM\left(w^2-1\right)}{\widetilde{\Theta}\left(ew^2-2w\log w-e\right)} \\
&= -\frac{\left(e^2-1\right)mM}{e\widetilde{\Theta}} \,.
\end{aligned} \tag{139}$$

At $t = T$ we have $u = 0$. Therefore:

$$\left.\frac{\partial x_{1g}}{\partial g}\right|_{t=T} = -x_2\left(e, \widetilde{\Theta}, 0\right) = 0 \tag{140}$$

Furthermore, $\partial x_{1g}/\partial g$ is asymptotically proportional to t at $t = -\infty$:

$$
\begin{aligned}
\lim_{t\to-\infty} \frac{\dfrac{\partial x_{1g}}{\partial g}}{t_p} &= \lim_{t\to-\infty} \frac{-x_2}{t_p} \\
&= \lim_{w\to-\infty} -\frac{\left(e^2-1\right) mM \left(w^2-1\right)}{\widetilde{\Theta}\left(ew^2+2w\log w-e\right)} \\
&= -\frac{\left(e^2-1\right) mM}{e\widetilde{\Theta}} \,. \qquad (141)
\end{aligned}
$$

The derivative of $x_1\left(e,\widetilde{\Theta},u\right)$ with respect to $\widetilde{\Theta}$ is:

$$
\frac{\partial x_1}{\partial \widetilde{\Theta}}\left(e,\widetilde{\Theta},u\right) = \frac{2\widetilde{\Theta}(e-\cosh u)}{\left(e^2-1\right) m^2 M} \,. \qquad (142)
$$

Thus, the derivative of $x_1\left(t,T,e,\widetilde{\Theta}\right)$ with respect to $\widetilde{\Theta}$ is:

$$
\begin{aligned}
\frac{\partial x_1}{\partial \widetilde{\Theta}}\left(t,T,e,\widetilde{\Theta}\right) &= \frac{\partial x_1}{\partial \widetilde{\Theta}}\left(e,\widetilde{\Theta},u\right) + \frac{\partial x_1}{\partial u}\frac{\partial u}{\partial \widetilde{\Theta}} \\
&= \frac{2\widetilde{\Theta}(e-\cosh u)}{\left(e^2-1\right) m^2 M} + \frac{3\sqrt{e^2-1}mM(t-T)\sinh u}{\widetilde{\Theta}^2(e\cosh u-1)} \,.
\end{aligned}
\qquad (143)
$$

The derivative $\partial x_1/\partial\widetilde{\Theta}$ is asymptotically proportional to t at $t = +\infty$:

$$\lim_{t\to+\infty}\frac{\frac{\partial x_1}{\partial\widetilde{\Theta}}}{t_p} = \lim_{w\to+\infty}\left(\frac{2\left(e^2-1\right)^{3/2}m^3M^2w}{\widetilde{\Theta}^3\left(ew^2-2w\log w-e\right)}\right.$$
$$\left.\left(\frac{3\sqrt{e^2-1}mM(t-T)\left(w^2-1\right)}{\left(ew^2-2w+e\right)\widetilde{\Theta}^2}-\frac{w^2\widetilde{\Theta}-2e\widetilde{\Theta}w+\widetilde{\Theta}}{(e^2-1)m^2Mw}\right)\right)$$
$$= -\frac{2\sqrt{e^2-1}mM}{e\widetilde{\Theta}^2}. \tag{144}$$

At $t = T$ we have $u = 0$. Therefore:

$$\left.\frac{\partial x_1}{\partial\widetilde{\Theta}}\right|_{t=T} = \frac{2\widetilde{\Theta}}{(e+1)m^2M} \tag{145}$$

Furthermore, $\partial x_1/\partial\widetilde{\Theta}$ is asymptotically proportional to t at $t = -\infty$:

$$\lim_{t\to-\infty}\frac{\frac{\partial x_1}{\partial\widetilde{\Theta}}}{t_p} = \lim_{w\to-\infty}\left(\frac{2\left(e^2-1\right)^{3/2}m^3M^2w}{\widetilde{\Theta}^3\left(ew^2+2w\log w-e\right)}\right.$$
$$\left.\left(\frac{\left(w^2+2ew+1\right)\widetilde{\Theta}}{\left(e^2-1\right)m^2Mw}-\frac{3\sqrt{e^2-1}mM(t-T)\left(w^2-1\right)}{\left(ew^2+2w+e\right)\widetilde{\Theta}^2}\right)\right)$$
$$= \frac{2\sqrt{e^2-1}mM}{e\widetilde{\Theta}^2}. \tag{146}$$

7.3 Calculation of the Derivatives of x_2

The derivatives of x_2 with respect to the parameters T, e, and $\widetilde{\Theta}$ can all be calculated by means of the Chain Rule, using the following derivative of x_2 with respect to pseudo-time u:

$$\frac{\partial x_2}{\partial u}\left(e, \widetilde{\Theta}, u\right) = \frac{\widetilde{\Theta}^2 \cosh u}{\sqrt{e^2-1}m^2 M}\,. \tag{147}$$

Recall that u is an implicitly-defined function of t, T, e, and $\widetilde{\Theta}$. We will make the customary "abuse of notation" and write x_2 for both the function $x_2\left(t, T, e, \widetilde{\Theta}\right)$ and the function $x_2\left(e, \widetilde{\Theta}, u\right)$ (hopefully, our meaning will be clear from context). Thus, the derivative of $x_2\left(t, T, e, \widetilde{\Theta}\right)$ with respect to T is:

$$\frac{\partial x_2}{\partial T}\left(t, T, e, \widetilde{\Theta}\right) = \frac{\partial x_2}{\partial u}\frac{\partial u}{\partial T} = \frac{\left(e^2-1\right) mM \cosh u}{\widetilde{\Theta}(1-e\cosh u)}\,. \tag{148}$$

We claim that $\partial x_2/\partial T$ is asymptotic to a constant at $t=+\infty$. The limit is easier to calculate if we make the substitution $u = \log w$. We know that $u \to +\infty$ as $t \to +\infty$ (since u is a monotone increasing function of t, and in particular, u is asymptotic to $\log t$). It follows that $w = \exp u \to +\infty$ as $t \to +\infty$ as well (and in fact, w is asymptotic to t). Hence:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\partial x_2}{\partial T} &= \lim_{w\to+\infty} \frac{\left(e^2-1\right) mM \cosh\left(\log w\right)}{\widetilde{\Theta}(1-e\cosh\left(\log w\right))} \\
&= \lim_{w\to+\infty} -\frac{\left(e^2-1\right) mM \left(w^2+1\right)}{\left(ew^2-2w+e\right)\widetilde{\Theta}} \\
&= -\frac{\left(e^2-1\right) mM}{e\widetilde{\Theta}}\,.
\end{aligned} \tag{149}$$

Note that at $t=T$ we have $u=0$. Therefore:

$$\left.\frac{\partial x_2}{\partial T}\right|_{t=T} = -\frac{(e+1)mM}{\widetilde{\Theta}}\,. \tag{150}$$

We claim that $\partial x_2/\partial T$ is asymptotic to a constant at $t = -\infty$. The limit is easier to calculate if we make the substitution $u = -\log(-w)$. We know that $u \to -\infty$ as $t \to -\infty$ (since u is a monotone increasing function of t, and in particular, u is asymptotic to $-\log(-t)$). It follows that $w = -\exp(-u) \to -\infty$ as $t \to -\infty$ as well (and in fact, w is asymptotic to t). Hence:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\partial x_2}{\partial T} &= \lim_{w\to+\infty} \frac{\left(e^2-1\right) mM \cosh\left(-\log\left(-w\right)\right)}{\widetilde{\Theta}(1 - e\cosh\left(-\log\left(-w\right)\right))} \\
&= \lim_{w\to+\infty} -\frac{\left(e^2-1\right) mM \left(w^2+1\right)}{\left(ew^2+2w+e\right)\widetilde{\Theta}} \\
&= -\frac{\left(e^2-1\right) mM}{e\widetilde{\Theta}} . \qquad (151)
\end{aligned}$$

The derivative of $x_2\left(e, \widetilde{\Theta}, u\right)$ with respect to e is:

$$\frac{\partial x_2}{\partial e}\left(e, \widetilde{\Theta}, u\right) = -\frac{e\widetilde{\Theta}^2 \sinh u}{\left(e^2-1\right)^{3/2} m^2 M} . \qquad (152)$$

Thus, the derivative of $x_2\left(t, T, e, \widetilde{\Theta}\right)$ with respect to e is:

$$\begin{aligned}
\frac{\partial x_2}{\partial e}\left(t, T, e, \widetilde{\Theta}\right) &= \frac{\partial x_2}{\partial e}\left(e, \widetilde{\Theta}, u\right) + \frac{\partial x_2}{\partial u}\frac{\partial u}{\partial e} \\
&= \left(\left(3e\left(e^2-1\right)^{3/2} m^3 M^2 (t-T) + \left(1-2e^2\right)\widetilde{\Theta}^3 \sinh u\right)\cosh u \right. \\
&\quad \left. + e\widetilde{\Theta}^3 \sinh u\right) \Big/ \left(\left(e^2-1\right)^{3/2} m^2 M \widetilde{\Theta}(e\cosh u - 1)\right) . \qquad (153)
\end{aligned}$$

We claim that $\partial x_2/\partial e$ is asymptotically proportional to t at $t = +\infty$. More precisely, the ratio $\left(\partial x_2/\partial e\right)/t$ has a constant limit. For the denominator in this ratio, we use equation (107) for $t_p\left(e,\widetilde{\Theta},u\right)$, the time from perihelion. As usual, the limit is easier to calculate if we make the substitution $u=\log w$.

$$\begin{aligned}\lim_{t\to+\infty}\frac{\dfrac{\partial x_2}{\partial e}}{t_p} &= \lim_{w\to+\infty}\Bigg(\frac{2mMw}{\left(ew^2-2w+e\right)\widetilde{\Theta}^4\left(ew^2-2w\log w-e\right)} \\ &\quad \left(e\left(w^2-1\right)\widetilde{\Theta}^3+\left(w^2+1\right)\left(3e\left(e^2-1\right)^{3/2}m^3M^2(t-T)\right.\right. \\ &\quad \left.\left.+\frac{\left(1-2e^2\right)\left(w^2-1\right)\widetilde{\Theta}^3}{2w}\right)\right)\Bigg) \\ &= \frac{\left(1-2e^2\right)mM}{e^2\widetilde{\Theta}}\,. \end{aligned} \tag{154}$$

At $t=T$ we have $u=0$. Therefore:

$$\left.\frac{\partial x_2}{\partial e}\right|_{t=T}=0\,. \tag{155}$$

Furthermore, $\partial x_2/\partial e$ is asymptotically proportional to t at $t=-\infty$:

$$\begin{aligned}\lim_{t\to-\infty}\frac{\dfrac{\partial x_2}{\partial e}}{t_p} &= \lim_{w\to-\infty}-\Bigg(\frac{2mMw}{\left(ew^2+2w+e\right)\widetilde{\Theta}^4\left(ew^2+2w\log w-e\right)} \\ &\quad \left(e\left(w^2-1\right)\widetilde{\Theta}^3-\left(w^2+1\right)\left(3e\left(e^2-1\right)^{3/2}m^3M^2(t-T)\right.\right. \\ &\quad \left.\left.+\frac{\left(1-2e^2\right)\left(w^2-1\right)\widetilde{\Theta}^3}{2w}\right)\right)\Bigg) \\ &= \frac{\left(1-2e^2\right)mM}{e^2\widetilde{\Theta}}\,. \end{aligned} \tag{156}$$

Now, the derivative $\partial x_{2g}/\partial g$ is effectively x_1, as explained earlier. This, too, is asymptotically proportional to t at $t = +\infty$:

$$
\begin{aligned}
\lim_{t\to+\infty} \frac{\frac{\partial x_{2g}}{\partial g}}{t_p} &= \lim_{t\to+\infty} \frac{x_1}{t_p} \\
&= \lim_{w\to+\infty} \frac{\sqrt{e^2-1}mM\left(-w^2+2ew-1\right)}{e\left(w^2-1\right)\widetilde{\Theta}-2w\widetilde{\Theta}\log w} \\
&= -\frac{\sqrt{e^2-1}mM}{e\widetilde{\Theta}}\,. \qquad (157)
\end{aligned}
$$

At $t = T$ we have $u = 0$. Therefore:

$$
\left.\frac{\partial x_{2g}}{\partial g}\right|_{t=T} = x_1\left(e, \widetilde{\Theta}, 0\right) = \frac{\widetilde{\Theta}^2}{(e+1)m^2M}\,. \qquad (158)
$$

Furthermore, $\partial x_{2g}/\partial g$ is asymptotically proportional to t at $t = -\infty$:

$$
\begin{aligned}
\lim_{t\to-\infty} \frac{\frac{\partial x_{2g}}{\partial g}}{t_p} &= \lim_{t\to-\infty} \frac{x_1}{t_p} \\
&= \lim_{w\to-\infty} \frac{\sqrt{e^2-1}mM\left(w^2+2ew+1\right)}{e\left(w^2-1\right)\widetilde{\Theta}+2w\widetilde{\Theta}\log\left(-w\right)} \\
&= \frac{\sqrt{e^2-1}mM}{e\widetilde{\Theta}}\,. \qquad (159)
\end{aligned}
$$

The derivative of $x_2\left(e, \widetilde{\Theta}, u\right)$ with respect to $\widetilde{\Theta}$ is:

$$
\frac{\partial x_2}{\partial \widetilde{\Theta}}\left(e, \widetilde{\Theta}, u\right) = \frac{2\widetilde{\Theta}\sinh u}{\sqrt{e^2-1}m^2M}\,. \qquad (160)
$$

Thus, the derivative of $x_2\left(t, T, e, \widetilde{\Theta}\right)$ with respect to $\widetilde{\Theta}$ is:

$$\frac{\partial x_2}{\partial \widetilde{\Theta}}\left(t, T, e, \widetilde{\Theta}\right) = \frac{\partial x_2}{\partial \widetilde{\Theta}}\left(e, \widetilde{\Theta}, u\right) + \frac{\partial x_2}{\partial u}\frac{\partial u}{\partial \widetilde{\Theta}}$$

$$= -\frac{3\left(e^2-1\right) mM(t-T)\cosh u}{\widetilde{\Theta}^2(e\cosh u - 1)} + \frac{2\widetilde{\Theta}\sinh u}{\sqrt{e^2-1}m^2 M}\,. \tag{161}$$

The derivative $\partial x_2/\partial\widetilde{\Theta}$ is asymptotically proportional to t at $t = +\infty$:

$$\lim_{t\to+\infty}\frac{\frac{\partial x_2}{\partial \widetilde{\Theta}}}{t_p} = \lim_{w\to+\infty}\left(\frac{2\left(e^2-1\right)^{3/2} m^3 M^2 w}{\widetilde{\Theta}^3\left(ew^2 - 2w\log w - e\right)}\left(\frac{\left(w^2-1\right)\widetilde{\Theta}}{\sqrt{e^2-1}m^2 M w} - \frac{3\left(e^2-1\right) mM(t-T)\left(w^2+1\right)}{\left(ew^2-2w+e\right)\widetilde{\Theta}^2}\right)\right)$$

$$= \frac{2\left(e^2-1\right) mM}{e\widetilde{\Theta}^2}\,. \tag{162}$$

At $t = T$ we have $u = 0$. Therefore:

$$\left.\frac{\partial x_2}{\partial \widetilde{\Theta}}\right|_{t=T} = 0 \tag{163}$$

Furthermore, $\partial x_2/\partial\widetilde{\Theta}$ is asymptotically proportional to t at $t = -\infty$:

$$
\begin{aligned}
\lim_{t\to-\infty} \frac{\frac{\partial x_2}{\partial \widetilde{\Theta}}}{t_p} &= \lim_{w\to-\infty} \left(\frac{2\left(e^2-1\right)^{3/2} m^3 M^2 w}{\widetilde{\Theta}^3 \left(ew^2 + 2w\log w - e\right)} \right. \\
&\quad \left. \left(\frac{\left(w^2-1\right)\widetilde{\Theta}}{\sqrt{e^2-1}m^2 M w} - \frac{3\left(e^2-1\right) mM(t-T)\left(w^2+1\right)}{\left(ew^2+2w+e\right)\widetilde{\Theta}^2} \right) \right) \\
&= \frac{2\left(e^2-1\right) mM}{e\widetilde{\Theta}^2} \, . \qquad (164)
\end{aligned}
$$

7.4 Summary of Results

In this summary, we denote the matrix of four linearly independent solutions by $\mathbf{X}$; in other words, we write the four columns of the 2×4 matrix of partial derivatives of the general solution as:

$$
\frac{\partial\left(x_1, x_2\right)}{\partial\left(T, e, g, \widetilde{\Theta}\right)} = \mathbf{X} \, . \qquad (165)
$$

Recall that all of the solutions have a common factor matrix of the form:

$$
\Xi\left(g, \widetilde{\Theta}\right) = \begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sign}\widetilde{\Theta} \end{bmatrix} . \qquad (166)
$$

As $t \to +\infty$ we get:

$$\mathbf{X} = \Xi\left(g,\widetilde{\Theta}\right)\frac{mM}{e\widetilde{\Theta}}\begin{bmatrix} \sqrt{e^2-1} & \frac{3e^2-1}{e\sqrt{e^2-1}}t & \left(1-e^2\right)t & \frac{-2\sqrt{e^2-1}}{\widetilde{\Theta}}t \\ 1-e^2 & \frac{1-2e^2}{e}t & -\sqrt{e^2-1}t & \frac{2\left(e^2-1\right)}{\widetilde{\Theta}}t \end{bmatrix}. \tag{167}$$

In table form, the right-hand matrix above is:

	$\frac{\partial}{\partial T}$	$\frac{\partial}{\partial e}$	$\frac{\partial}{\partial g}$	$\frac{\partial}{\partial \widetilde{\Theta}}$
x_1	$\sqrt{e^2-1}$	$\frac{3e^2-1}{e\sqrt{e^2-1}}t$	$\left(1-e^2\right)t$	$\frac{-2\sqrt{e^2-1}}{\widetilde{\Theta}}t$.
x_2	$1-e^2$	$\frac{1-2e^2}{e}t$	$-\sqrt{e^2-1}t$	$\frac{2\left(e^2-1\right)}{\widetilde{\Theta}}t$

At $t = T$ we get:

$$\mathbf{X} = \Xi\left(g,\widetilde{\Theta}\right)\begin{bmatrix} 0 & -\frac{\widetilde{\Theta}^2}{(e+1)^2m^2M} & 0 & \frac{2\widetilde{\Theta}}{(e+1)m^2M} \\ -\frac{(e+1)mM}{\widetilde{\Theta}} & 0 & \frac{\widetilde{\Theta}^2}{(e+1)m^2M} & 0 \end{bmatrix}. \tag{168}$$

In table form, the right-hand matrix above is:

	$\dfrac{\partial}{\partial T}$	$\dfrac{\partial}{\partial e}$	$\dfrac{\partial}{\partial g}$	$\dfrac{\partial}{\partial \widetilde{\Theta}}$
x_1	0	$-\dfrac{\widetilde{\Theta}^2}{(e+1)^2 m^2 M}$	0	$\dfrac{2\widetilde{\Theta}}{(e+1)m^2 M}$
x_2	$-\dfrac{(e+1)mM}{\widetilde{\Theta}}$	0	$\dfrac{\widetilde{\Theta}^2}{(e+1)m^2 M}$	0

.

As $t \to -\infty$ we get:

$$\mathbf{X} \quad = \quad \Xi\left(g, \widetilde{\Theta}\right) \frac{mM}{e\widetilde{\Theta}} \begin{bmatrix} -\sqrt{e^2-1} & \dfrac{1-3e^2}{e\sqrt{e^2-1}}t & \left(1-e^2\right)t & \dfrac{2\sqrt{e^2-1}}{\widetilde{\Theta}}t \\ 1-e^2 & \dfrac{1-2e^2}{e}t & \sqrt{e^2-1}t & \dfrac{2\left(e^2-1\right)}{\widetilde{\Theta}}t \end{bmatrix}. \tag{169}$$

In table form, the right-hand matrix above is:

$$
\begin{array}{c|cccc}
 & \dfrac{\partial}{\partial T} & \dfrac{\partial}{\partial e} & \dfrac{\partial}{\partial g} & \dfrac{\partial}{\partial \widetilde{\Theta}} \\
\hline
x_1 & -\sqrt{e^2-1} & \dfrac{1-3e^2}{e\sqrt{e^2-1}}t & \left(1-e^2\right)t & \dfrac{2\sqrt{e^2-1}}{\widetilde{\Theta}}t \\
x_2 & 1-e^2 & \dfrac{1-2e^2}{e}t & \sqrt{e^2-1}t & \dfrac{2\left(e^2-1\right)}{\widetilde{\Theta}}t
\end{array}
.
$$

The $\pm\infty$ calculations again show that $\partial \mathbf{x}/\partial T$ is bounded and the solutions are linearly independent. The error terms are $\mathcal{O}\left(1/\log t\right)$ for the first column (the entries that are asymptotically constant) and are $\mathcal{O}\left(\log t\right)$ for the remaining columns (the entries that are asymptotically proportional to t). Interestingly, the (x_1, x_2) values (the rows in the matrix) are all the same as $t \to -\infty$ and as $t \to +\infty$ except for sign. But the signs do not change in the same way in each column (else we *would* have the possibility of a consistent linear dependency). Of the three unbounded columns (asymptotic to a multiple of t at infinity), in two of them the sign of the x_1 component is opposite between $+\infty$ and $-\infty$, while the sign of the x_2 component stays the same; the third one is the opposite, the sign of the x_1 component stays the same and the sign of the x_2 component changes. But by these calculations, we have proved more than the linear independence of the four solutions. We have proved the following:

Theorem 7.1: In the kernel of the linearized Kepler operator, hyperbolic case, there is exactly one bounded function (up to multiplication by a scalar constant), namely $\partial \mathbf{x}/\partial T$. Any linear combination of the other three (linearly independent) functions that span this kernel produces a function that is unbounded at either

$-\infty$ or $+\infty$ (or both). More precisely, the unbounded solutions are asymptotically proportional to t at either or both $-\infty$ or $+\infty$.

Proof: Since $\partial\mathbf{x}/\partial e$, $\partial\mathbf{x}/\partial g$, and $\partial\mathbf{x}/\partial\widetilde{\Theta}$ are all asymptotically proportional to t, a linear combination of them would be bounded only if it vanishes at *both* $+\infty$ and $-\infty$. That is, (omitting common factors) the vectors

$$\mathbf{v}_2 = \begin{bmatrix} \dfrac{3e^2-1}{e\sqrt{e^2-1}} \\ \\ \dfrac{1-2e^2}{e} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} (1-e^2) \\ \\ -\sqrt{e^2-1} \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} \dfrac{-2\sqrt{e^2-1}}{\widetilde{\Theta}} \\ \\ \dfrac{2\,(e^2-1)}{\widetilde{\Theta}} \end{bmatrix} \tag{170}$$

as $t \to +\infty$, and

$$\mathbf{w}_2 = \begin{bmatrix} \dfrac{1-3e^2}{e\sqrt{e^2-1}} \\ \\ \dfrac{1-2e^2}{e} \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} (1-e^2) \\ \\ \sqrt{e^2-1} \end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix} \dfrac{2\sqrt{e^2-1}}{\widetilde{\Theta}} \\ \\ \dfrac{2\,(e^2-1)}{\widetilde{\Theta}} \end{bmatrix} \tag{171}$$

as $t \to -\infty$, would have to admit the *same* linear dependency. First, note that any linear dependency of the first three vectors above, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, must necessarily require a non-zero scalar coefficient of the middle vector, $\mathbf{v}_3$. That is because $\mathbf{v}_2$ and $\mathbf{v}_4$ are in fact linearly independent, as shown by

$$\det \begin{bmatrix} \dfrac{3e^2-1}{e\sqrt{e^2-1}} & -\dfrac{2\sqrt{e^2-1}}{\Theta} \\ \\ \dfrac{1-2e^2}{e} & \dfrac{2\,(e^2-1)}{\Theta} \end{bmatrix} = \frac{2e\sqrt{e^2-1}}{\Theta} \neq 0\,. \tag{172}$$

Now suppose that we have a linear dependency of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ (such a dependency surely exists, because they are three vectors in $\mathbb{R}^2$). Then we have three scalars α, β, and γ, with $\beta \neq 0$, such that

$$\alpha \mathbf{v}_2 + \beta \mathbf{v}_3 + \gamma \mathbf{v}_4 = 0\,. \tag{173}$$

Let a, b, and c stand for the first components of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, respectively (that is, $\mathbf{v}_2 = (a, \ldots)$, $\mathbf{v}_3 = (b, \ldots)$, and $\mathbf{v}_4 = (c, \ldots)$). Then, in particular, we have $\alpha a + \beta b + \gamma c = 0$ from the linear dependency. But the first components of $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ are $-a$, b, and $-c$, respectively (that is, $\mathbf{w}_2 = (-a, \ldots)$, $\mathbf{w}_3 = (b, \ldots)$, and $\mathbf{w}_4 = (-c, \ldots)$), and if $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ satisfied the same linear dependency as $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, then we would also have $\alpha\,(-a) + \beta b + \gamma\,(-c) = 0$. That is not possible, because

$$\begin{aligned} \alpha\,(-a) + \beta b + \gamma\,(-c) &= -\,(\alpha a + \beta b + \gamma c) + 2\beta b \\ &= 0 + 2\beta b \\ &= 2\beta b \\ &= 2\beta\left(1 - e^2\right) \neq 0\,. \end{aligned} \tag{174}$$

■

8. Fredholm Properties of the Linearized Kepler Operator

Notation: In the following discussion, let $\mathcal{R}(f)$ denote the range of the function f. (The functions that we shall be discussing will, in general, be linear operators.)

Definition: Let B_1 and B_2 be real Banach spaces (or possibly Hilbert spaces), and let $F : B_1 \to B_2$ be a linear operator. Then F is called a *Fredholm operator* provided that it satisfies the following three conditions:

1. The kernel (also known as "null space") of F is finite-dimensional.

2. The cokernel of F is finite-dimensional (*i.e.*, $B_2/\mathcal{R}(F)$ is finite-dimensional). This property can also be stated as "$\mathcal{R}(F)$ has finite codimension in B_2".

3. $\mathcal{R}(F)$ is closed in B_2.

Notes:

- A short exact sequence of vector spaces, in which at least one space is finite-dimensional, always splits topologically. So condition 2 is equivalent to $B_2 = \mathcal{R}(F) \oplus M$, where M is finite-dimensional. The complementary subspace M is not unique. But when B_2 is a Hilbert space, a good choice for M is $\mathcal{R}(F)^{\perp}$.

- Property 3 in the above definition is redundant. If follows automatically from properties 1 and 2, by application of the Open Mapping Theorem.

- A standard fact from functional analysis is that an operator is Fredholm if and only if the following (alternate definition) holds: A Fredholm operator is a bounded linear operator $F : B_1 \to B_2$ that has left and right inverses modulo compact operators. That is, there exist bounded linear operators $G_1, G_2 : B_2 \to B_1$ such that $G_1 \circ F = I_{B_1} - K_1$ and $F \circ G_2 = I_{B_2} - K_2$, where K_1 and K_2 are compact operators. (See [**5**].)

- In many applications, B_1 is a linear subspace of B_2. It may happen that B_1 is dense in B_2. If B_1 is a proper dense subset of B_2, then it cannot be closed (else it would not be proper), hence it is not complete. In such a case, B_1 is therefore not a Banach (or Hilbert) space with respect to $\|\cdot\|_{B_2}$.

Example 8.1: Let H be a Hilbert space and let $K : H \to H$ be a compact operator. Then by the Fredholm Alternative Theorem, $K + \mu I$, where $\mu \in \mathbb{R}$, $\mu \neq 0$, satisfies the three properties above and $K + \mu I$ is thus a Fredholm operator. Hence, the definition of a Fredholm operator can be seen as a generalization of the properties of operators of the form $K + I$, where K is compact.

Example 8.2: Let $H = l^2(\mathbb{R})$ and define $F : (a_1, a_2, a_3, \ldots) \mapsto (a_2, a_3, a_4, \ldots)$, *i.e.*, drop a_1 and shift the remaining elements of the square-summable sequence one position to the left. Then the kernel of F is the one-dimensional subspace spanned by $(1, 0, 0, \ldots)$ and $\mathcal{R}(F) = H$, so the cokernel is trivial (zero-dimensional) and thus F satisfies the definition of a Fredholm operator. However, F is not of the form $K + \mu I$, where K is a compact operator and $\mu \in \mathbb{R}$, $\mu \neq 0$, for if F were of that form, then by the Fredholm Alternative Theorem, the kernel and cokernel of F would have the same dimension.

Definition: Let B_1 and B_2 be real Banach spaces, and let $F : B_1 \to B_2$ be a Fredholm operator. Then the *index* of F, denoted by $\operatorname{ind} F$, is defined by $\operatorname{ind} F = \dim(\ker F) - \dim(\operatorname{coker} F)$.

Multiplication Theorem for Fredholm Operators: Let B_1, B_2, and B_3 be real Banach spaces, and let $F : B_1 \to B_2$ and $G : B_2 \to B_3$ be Fredholm operators. Then $G \circ F : B_1 \to B_3$ is also a Fredholm operator. Furthermore, the index of $G \circ F$ is the sum of the index of F and the index of G.

Proof: This is a well-known result. The proof is a simple argument involving only elementary finite-dimensional linear algebra. See [**14**].

■

Compact Perturbation Theorem: Let B_1 and B_2 be real Banach spaces, let $F : B_1 \to B_2$ be a Fredholm operator, and let $K : B_1 \to B_2$ be a compact operator. Then $G = K + \mu F : B_1 \to B_2$, where $\mu \in \mathbb{R}$, $\mu \neq 0$, is also a Fredholm operator. Furthermore, the index of G is the same as the index of F. (This can also be stated as "The index of F remains constant under compact perturbations of F".)

Proof: This is a well-known result. The proof is a generalization of the proof of the Fredholm Alternative Theorem. See [**5**].

■

Definition: For $t \in \mathbb{R}$ we define $\langle t \rangle = \sqrt{1+t^2}$. Asymptotically, $\langle t \rangle \sim |t|$ as $t \to \pm\infty$, but $\langle 0 \rangle = 1$. We will use $\langle t \rangle$ instead of $|t|$ so that the function has a positive lower bound (namely 1). In particular, this avoids "divide by zero" issues when raising $\langle t \rangle$ to a negative power.

In the following, we shall take C^k to be the space of all k-times continuously differentiable functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$. Typically, we will be dealing with $n = 2$; in other words, C^k curves in the plane defined for all time $t \in \mathbb{R}$.

Definition: Let $k \in \mathbb{Z}, k \geq 0$, and let $\delta \in \mathbb{R}$. Define C^k_δ, the *δ-weighted spaces* of C^k functions, by

$$C^0_\delta = \left\{ \mathbf{f} \in C^0 : \|\mathbf{f}(t)\| \leq c \langle t \rangle^\delta, \text{ for some } c > 0 \right\},$$

$$C^1_\delta = \left\{ \mathbf{f} \in C^1 : \mathbf{f} \in C^0_\delta \text{ and } \|\mathbf{f}'(t)\| \leq c \langle t \rangle^{\delta-1}, \text{ for some } c > 0 \right\},$$

$$C^2_\delta = \left\{ \mathbf{f} \in C^2 : \mathbf{f} \in C^1_\delta \text{ and } \|\mathbf{f}''(t)\| \leq c \langle t \rangle^{\delta-2}, \text{ for some } c > 0 \right\},$$

$$\vdots$$

$$C^k_\delta = \left\{ \mathbf{f} \in C^k : \mathbf{f} \in C^{k-1}_\delta \text{ and } \left\|\mathbf{f}^{(k)}(t)\right\| \leq c \langle t \rangle^{\delta-k}, \text{ for some } c > 0 \right\}.$$

We will refer to the inequalities that appear in this definition as *bounding envelope inequalities.*

Notes:

- In particular, C_0^0 is the set of bounded continuous functions, and C_0^k is the set of functions $\mathbf{f} \in C^k$ such that for $j = 0, \ldots, k$, the function $\mathbf{f}^{(j)}$ is bounded by the function $c_j \langle t \rangle^{-j}$, for some $c_j > 0$.

- A necessary and sufficient condition for $\mathbf{f}$ to belong to C_δ^k is that $\mathbf{f}(t) \langle t \rangle^{-\delta} = \mathbf{g}(t) \in C_0^k$. Necessity is easily shown by using the Leibniz product rule to differentiate $\mathbf{g}(t) = \mathbf{f}(t) \langle t \rangle^{-\delta}$, repeatedly, and simplifying the resulting inequalities. Note that by using the Chain Rule and the fact that

$$\left| \frac{d}{dt} \langle t \rangle \right| = \left| \frac{t}{\sqrt{1+t^2}} \right| = \sqrt{\frac{t^2}{t^2+1}} < 1, \tag{175}$$

one can readily show that

$$\left| \frac{d^j}{dt^j} \langle t \rangle^{-\delta} \right| < c \left| \langle t \rangle^{-\delta-j} \right|, \tag{176}$$

where $c > 0$ is a constant that depends only on δ and j.

Sufficiency is shown by applying the same reasoning to $\mathbf{f}(t) = \mathbf{g}(t) \langle t \rangle^{\delta}$.

- When $\delta < \delta'$, it is clear that $C_\delta^k \subset C_{\delta'}^k$.

- There is a broad "family" of theories of δ-weighted spaces, in which the weight factor varies according to a parameter $\delta \in \mathbb{R}$. There are a variety of spaces considered, including Hilbert spaces, Sobolev spaces, and (as in the present instance) C^k spaces. There are also different weight factors that can be employed, including exponential weights and monomial weights (as are used here).

Definition: For $\mathbf{f} \in C_\delta^k$, we define $\|\mathbf{f}\|_\delta^k$ as follows. First, we define $c_0, c_1, c_2, \ldots, c_k$ by:

$$c_0 = \inf \left\{ c : \|\mathbf{f}(t)\| \leq c \langle t \rangle^\delta \text{ for all } t \in \mathbb{R} \right\},$$

$$c_1 = \inf \left\{ c : \|\mathbf{f}'(t)\| \leq c \langle t \rangle^{\delta-1} \text{ for all } t \in \mathbb{R} \right\},$$

$$c_2 = \inf \left\{ c : \|\mathbf{f}''(t)\| \leq c \langle t \rangle^{\delta-2} \text{ for all } t \in \mathbb{R} \right\},$$

$$\vdots$$

$$c_k = \inf \left\{ c : \left\|\mathbf{f}^{(k)}(t)\right\| \leq c \langle t \rangle^{\delta-k} \text{ for all } t \in \mathbb{R} \right\}.$$

Then we define $\|\mathbf{f}\|_\delta^k = \max\{c_0, c_1, c_2, \ldots, c_k\}$. (We could just as well use $\sum_{j=0}^k c_j$ or $\sqrt{c_0^2 + c_1^2 + \cdots + c_k^2}$.)

Notes:

- In particular, $\|\cdot\|_0^0$ is simply the uniform norm on C_0^0, the space of bounded continuous functions.
- The C_δ^k spaces are complete under the $\|\cdot\|_\delta^k$ norm, hence they are Banach spaces. (That follows from elementary analysis, the properties of uniformly convergent sequences of C^k functions.)

Theorem 8.1: Let D be the differentiation operator, and let $k \in \mathbb{Z}, k > 0$. Then $D : C_\delta^k \to C_{\delta-1}^{k-1}$ is a Fredholm operator, except when $\delta = 0$.

Proof: First, note that if $\mathbf{f} \in C_\delta^k$, then $Df = \mathbf{g} \in C^{k-1}$ and the given bounding envelope inequalities in C_δ^k for $\mathbf{g} = \mathbf{f}', \mathbf{g}' = \mathbf{f}'', \ldots, \mathbf{g}^{(k-1)} = \mathbf{f}^{(k)}$ are precisely the inequalities that we need to establish that $\mathbf{g} \in C_{\delta-1}^{k-1}$. So it is clear that $D : C_\delta^k \to C_{\delta-1}^{k-1}$.

The kernel of D (on C^k) consists of the constant functions. The non-zero constant functions $\mathbf{f}(t) = c$ are *not* in C_δ^k for $\delta < 0$ (since $\mathbf{f}(t) \langle t \rangle^{-\delta} = c \langle t \rangle^{-\delta}$ is unbounded in that case, hence $\mathbf{f}(t) \langle t \rangle^{-\delta} \notin C_0^k$). But the constant functions $\mathbf{f}(t) = c$ *are* in C_δ^k for $\delta \geq 0$ (since $\mathbf{f}(t) \langle t \rangle^{-\delta} = c \langle t \rangle^{-\delta}$ is bounded, hence $\mathbf{f}(t) \langle t \rangle^{-\delta} \in C_0^1$; and $\mathbf{f}'(t) \langle t \rangle^{-\delta} = 0$, which is certainly bounded by $\langle t \rangle^{-1}$). Since $\mathbb{R}^n$ is the codomain of our functions, the dimension of the subspace of constant functions is n. So the dimension of the kernel of D is either 0 or n, which is finite in either case.

Now consider a function $\mathbf{g} \in C_{\delta-1}^{k-1}$. If $\mathbf{g} \in \mathcal{R}(D)$, then there exists some $\mathbf{f} \in C_\delta^k$ such that $Df = \mathbf{g}$. Clearly, any such function $\mathbf{f}$ must be an indefinite integral of $\mathbf{g}$, and it must satisfy the prescribed bounding envelope inequality in C_δ^k. The calculation of the bounding properties of $\int \mathbf{g}$ depends on the value of δ. Before we give those calculations, we need to establish a set of inequalities for the function $\langle t \rangle = \sqrt{1+t^2}$. First, note that both of the following inequalities hold for all $t \in \mathbb{R}$:

$$\langle t \rangle \geq 1, \tag{177}$$

$$\langle t \rangle \geq |t| \ . \tag{178}$$

Furthermore, when $|t| \leq 1$ we have:

$$\langle t \rangle \leq \sqrt{2} \, . \tag{179}$$

On the other hand, when $|t| \geq 1$ we have:

$$\langle t \rangle \leq \sqrt{2} \, |t| \ . \tag{180}$$

Since $\mathbf{g} \in C^{k-1}_{\delta-1}$, it satisfies a bounding inequality of the form $\|\mathbf{g}(t)\| \leq c \langle t \rangle^{\delta-1}$. Now, there are three cases to consider. First, we consider the case that $\delta \geq 1$, in which case $\delta - 1 \geq 0$, so the above inequalities are preserved when raised to the $\delta - 1$ power, as well as when raised to the δ power. Let $\mathbf{f} \in C^k$ be defined by $\mathbf{f}(t) = \int_0^t \mathbf{g}$. Then we have:

$$\|\mathbf{f}(t)\| = \left\|\int_0^t \mathbf{g}\right\| \leq \int_0^{|t|} \|\mathbf{g}\| \leq c\int_0^{|t|} \langle\tau\rangle^{\delta-1}\,d\tau$$

$$\leq \begin{cases} c\left(\sqrt{2}\right)^{\delta-1} & \text{when } |t| \leq 1 \\ c\left(\sqrt{2}\right)^{\delta-1} + c\int_1^{|t|}\left(\sqrt{2}\,|\tau|\right)^{\delta-1}d\tau & \text{when } |t| \geq 1 \end{cases}$$

$$= \begin{cases} c\left(\sqrt{2}\right)^{\delta-1} & \text{when } |t| \leq 1 \\ c\left(\sqrt{2}\right)^{\delta-1} + \dfrac{c\left(\sqrt{2}\right)^{\delta-1}}{\delta}|t|^{\delta} - \dfrac{c\left(\sqrt{2}\right)^{\delta-1}}{\delta} & \text{when } |t| \geq 1 \end{cases}$$

$$\leq c\left(\sqrt{2}\right)^{\delta-1}\langle t\rangle^{\delta} + c\left(\sqrt{2}\right)^{\delta-1}\langle t\rangle^{\delta} = 2c\left(\sqrt{2}\right)^{\delta-1}\langle t\rangle^{\delta}, \tag{181}$$

so $\mathbf{f}$ satisfies the required bounding inequality for a function in C_δ^k, and the given bounding envelope inequalities in $C_{\delta-1}^{k-1}$ for $\mathbf{f}' = \mathbf{g}, \mathbf{f}'' = \mathbf{g}', \ldots, \mathbf{f}^{(k)} = \mathbf{g}^{(k-1)}$ are precisely the remaining inequalities that we need to establish that $\mathbf{f} \in C_\delta^k$. So it is clear that when $\delta \geq 1$, D is surjective and thus $\mathcal{R}(D) = C_{\delta-1}^{k-1}$.

The second case is when $0 < \delta < 1$, in which case $\delta - 1 < 0$, so the above inequalities are reversed when raised to the $\delta - 1$ power, but they are preserved when raised to the δ power. Again let $\mathbf{f} \in C^k$ be defined by $\mathbf{f}(t) = \int_0^t \mathbf{g}$. Then we have:

$$\|\mathbf{f}(t)\| = \left\| \int_0^t \mathbf{g} \right\| \leq \int_0^{|t|} \|\mathbf{g}\| \leq c \int_0^{|t|} \langle \tau \rangle^{\delta-1} \, d\tau$$

$$\leq \begin{cases} c\,(1)^{\delta-1} & \text{when } \ |t| \leq 1 \\ \\ c\,(1)^{\delta-1} + c \int_1^{|t|} |\tau|^{\delta-1} \, d\tau & \text{when } \ |t| \geq 1 \end{cases}$$

$$= \begin{cases} c & \text{when } \ |t| \leq 1 \\ \\ c + \dfrac{c}{\delta} |t|^{\delta} - \dfrac{c}{\delta} & \text{when } \ |t| \geq 1 \end{cases}$$

$$\leq \frac{c}{\delta} \langle t \rangle^{\delta} + \frac{c}{\delta} \langle t \rangle^{\delta} = 2\frac{c}{\delta} \langle t \rangle^{\delta}, \tag{182}$$

so, again, $\mathbf{f}$ satisfies the required bounding inequality for a function in C_δ^k, and the given bounding envelope inequalities for $\mathbf{g} \in C_{\delta-1}^{k-1}$ are precisely the remaining inequalities that we need to establish that $\mathbf{f} \in C_\delta^k$. So, when $0 < \delta < 1$, D is surjective, just as it is for $\delta \geq 1$; thus, for *all* $\delta > 0$ we have $\mathcal{R}(D) = C_{\delta-1}^{k-1}$, and consequently $\dim(\operatorname{coker} D) = 0$.

The third (and final) case is when $\delta < 0$, in which case *both* $\delta - 1$ and δ are negative, so the above inequalities are reversed when raised to both the $\delta - 1$ power and the δ power. In this case, the integral $\int_0^t \mathbf{g}$ does *not* give us a function in C_δ^k, because when $\delta < 0$, the "bounding envelope" curves downward, converging to a norm of 0 at both $-\infty$ and $+\infty$ infinity, and (in general) the function $\mathbf{f}(t) = \int_0^t \mathbf{g}$ fails to have this property. In fact, the calculation of $\|\mathbf{f}(t)\|$ proceeds exactly as it did for $0 < \delta < 1$, and for $|t| \geq 1$ we get

$$\left\| \int_0^t \mathbf{g} \right\| \leq c + \frac{c}{\delta} |t|^\delta - \frac{c}{\delta} = c + \frac{c}{|\delta|} - \frac{c}{|\delta|} |t|^\delta , \tag{183}$$

which we have re-written using $|\delta| = -\delta$, because (in the present case) δ is negative. This bound actually *increases* from a value of c at $|t| = 1$ to a value of $c(|\delta| + 1) / |\delta|$ as $|t| \to \infty$. That does not prove that $\int_0^t \mathbf{g}$ fails to have the bounds that we seek, but it strongly suggests that we should (at least) use a different lower-limit for our indefinite integral. In fact, it is clear that we must take $\mathbf{f}(t) = \int_{-\infty}^t \mathbf{g}$, so that $\lim_{t \to -\infty} \|\mathbf{f}(t)\| = 0$. But we also require $\lim_{t \to +\infty} \|\mathbf{f}(t)\| = 0$, and so we must also take $\mathbf{f}(t) = \int_{+\infty}^t \mathbf{g}$. Therefore we require both choices for $\mathbf{f}$ to agree:

$$\mathbf{f}(t) = \int_{-\infty}^t \mathbf{g} = \int_{+\infty}^t \mathbf{g}, \tag{184}$$

and consequently:

$$\int_{-\infty}^t \mathbf{g} - \int_{+\infty}^t \mathbf{g} = (0, \ldots, 0), \tag{185}$$

$$\int_{-\infty}^t \mathbf{g} + \int_t^{+\infty} \mathbf{g} = (0, \ldots, 0), \tag{186}$$

$$\int_{-\infty}^{+\infty} \mathbf{g} = (0, \ldots, 0) . \tag{187}$$

Now we must check that $\mathbf{f}(t) = \int_{-\infty}^{t} \mathbf{g}$ actually satisfies the required bounding inequality. For $t \leq -1$, we have:

$$
\begin{aligned}
\|\mathbf{f}(t)\| &= \left\| \int_{-\infty}^{t} \mathbf{g} \right\| \leq \int_{-\infty}^{t} \|\mathbf{g}\| \leq c \int_{-\infty}^{t} \langle \tau \rangle^{\delta - 1} d\tau \\
&\leq c \int_{-\infty}^{t} |\tau|^{\delta - 1} d\tau = c \int_{|t|}^{+\infty} \tau^{\delta - 1} d\tau \\
&= -\frac{c}{\delta} |t|^{\delta} = \frac{c}{|\delta|} |t|^{\delta} \\
&\leq \frac{c}{|\delta| \left(\sqrt{2}\right)^{\delta}} \langle t \rangle^{\delta} .
\end{aligned} \tag{188}
$$

Similarly, using $\mathbf{f}(t) = \int_{+\infty}^{t} \mathbf{g}$ and assuming that $t \geq 1$, we have:

$$
\begin{aligned}
\|\mathbf{f}(t)\| &= \left\| \int_{+\infty}^{t} \mathbf{g} \right\| \leq \int_{t}^{+\infty} \|\mathbf{g}\| \leq c \int_{t}^{+\infty} \langle \tau \rangle^{\delta - 1} d\tau \\
&\leq \frac{c}{|\delta| \left(\sqrt{2}\right)^{\delta}} \langle t \rangle^{\delta} ,
\end{aligned} \tag{189}
$$

and, furthermore, $\|\mathbf{f}\|$ is bounded on the compact interval $[-1, 1]$. So, subject to the constraint that $\int_{-\infty}^{+\infty} \mathbf{g} = (0, \ldots, 0)$, this choice of $\mathbf{f}$ satisfies the required bounding inequality for a function in C_{δ}^{k}, and the given bounding envelope inequalities for $\mathbf{g} \in C_{\delta-1}^{k-1}$ are precisely the remaining inequalities that we need to establish that $\mathbf{f} \in C_{\delta}^{k}$.

Of course, not all functions $\mathbf{g} \in C^{k-1}_{\delta-1}$ satisfy the constraint that $\int_{-\infty}^{+\infty} \mathbf{g} = (0, \ldots, 0)$. In fact, $F(\mathbf{g}) = \int_{-\infty}^{+\infty} \mathbf{g}$ is a linear operator, such that $F : C^{k-1}_{\delta-1} \to \mathbb{R}^n$, and the range of the D operator is precisely the kernel of F. The linear operator F is clearly surjective, since the C^∞ function

$$\mathbf{g}(t) = \frac{1}{\sqrt{\pi}} \left(a_1 e^{-t^2}, a_2 e^{-t^2}, \ldots, a_n e^{-t^2} \right) \tag{190}$$

belongs to *all* of the $C^{k-1}_{\delta-1}$ spaces, and $F(\mathbf{g}) = (a_1, a_2, \ldots, a_n)$. It follows, therefore, that the cokernel of D, which is the quotient of $C^{k-1}_{\delta-1}$ by the kernel of F, is isomorphic to $\mathbb{R}^n$. Thus, the cokernel of D is n-dimensional. This argument also shows that $\mathcal{R}(D)$ is closed, as it is the inverse image of the closed set $\{(0, \ldots, 0)\}$ under the continuous function F.

In summary, we have shown that when $\delta > 0$, the D operator is surjective, hence $\dim(\operatorname{coker} D) = 0$; but in the case that $\delta < 0$, the D operator is *not* surjective, and in fact in that case $\dim(\operatorname{coker} D) = n$.

■

Comment: Why did we exclude $\delta = 0$ in the above theorem? The answer lies in the fact that $\int \frac{1}{t} dt = \log t$. In particular, the function $\langle t \rangle^{-1}$ is in C^{k-1}_{-1}, yet its antiderivative, which is (approximately) $\log \langle t \rangle$, is unbounded and hence does not belong to C^k_0. But $\langle t \rangle^{-1}$ is the uniform limit of the sequence of functions

$$\left\{ \langle t \rangle^{-1-\frac{1}{j}} \right\}_j , \tag{191}$$

all of which lie in C^{k-1}_{-1} and all of which have antiderivatives that belong to C^k_0. Hence $\mathcal{R}(D)$ is not closed when the domain of D is C^k_0. Note that this also implies that the cokernel of D must be infinite-dimensional in this case, for if it were finite-dimensional then $\mathcal{R}(D)$ would necessarily be closed (by the Open Mapping Theorem).

Definition: Weight parameters δ for which an operator $F : C^k_\delta \to C^{k-1}_{\delta-1}$ fails to be Fredholm are called *indicial points*. This terminology is due to the fact that, for differential operators such as D, these points are the roots of the *indicial equation* that arises in the Frobenius method for solving ordinary differential equations (ODEs) by means of power series. (See **[7]** and **[3]**.)

Corollary 8.2: $D^k : C^k_\delta \to C^0_{\delta-k}$ is a Fredholm operator, except when $\delta = 0, 1, \ldots k-1$.

Proof: $D^k = D \circ D \circ \cdots \circ D : C^k_\delta \to C^{k-1}_{\delta-1} \to C^{k-2}_{\delta-2} \to \cdots \to C^0_{\delta-k}$. (Compositions of Fredholm operators are Fredholm, see the Multiplication Theorem for Fredholm Operators, stated above.) We get multiple indicial points because the successive mappings drop the weight parameter δ in the following sequence: $\delta \mapsto \delta - 1 \mapsto \cdots \mapsto \delta - k + 1 \mapsto \delta - k$. Hence, the first D operator introduces an indicial point at $\delta = 0$, the second D operator introduces an indicial point at $\delta - 1 = 0$, and so on, until the final D operator introduces an indicial point at $\delta - k + 1 = 0$.

■

Note: For the composite Fredholm operator $D^k : C^k_\delta \to C^0_{\delta-k}$, the index values of the composite operators add, in accordance with the Multiplication Theorem for Fredholm Operators, as shown in the following table:

$D^k : C^k_\delta \to C^0_{\delta-k}$	$\delta < 0$	$0 < \delta < 1$	$\cdots$	$k-2 < \delta < k-1$	$k-1 < \delta$
$\operatorname{ind} D$	$-n$	n	$\cdots$	n	n
$\operatorname{ind} D$	$-n$	$-n$	$\cdots$	n	n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\operatorname{ind} D$	$-n$	$-n$	$\cdots$	$-n$	n
$\operatorname{ind} D^k$	$-kn$	$-(k-2)\,n$	$\cdots$	$(k-2)\,n$	kn

The dimensions of the kernel and cokernel "build up", in the opposite order from each other. For $\delta > k-1$, $\dim\left(\ker D^k\right) = kn$, since $\ker D^k$ consists of the n-tuples of polynomials of degree $k-1$. For $k-2 < \delta < k-1$, $\dim\left(\ker D^k\right) = (k-1)\,n$, and $\dim\left(\ker D^k\right)$ continues to decrease in that fashion between each pair of indicial points, until $\dim\left(\ker D^k\right) = n$ for $0 < \delta < 1$ and $\dim\left(\ker D^k\right) = 0$ for $\delta < 0$. But $\dim(\operatorname{coker} F) = \dim(\ker F) - \operatorname{ind} F$. Consequently, the dimensions of the cokernel follow the same sequence in the opposite order, with $\dim\left(\operatorname{coker} D^k\right) = kn$ for $\delta < 0$ and $\dim\left(\operatorname{coker} D^k\right) = 0$ for $\delta > k-1$. The following table illustrates this:

$D^k : C^k_\delta \to C^0_{\delta-k}$	$\delta < 0$	$0 < \delta < 1$	$\cdots$	$k-2 < \delta < k-1$	$k-1 < \delta$
dim(kernel)	0	n	$\cdots$	$(k-1)\,n$	kn
dim(cokernel)	kn	$(k-1)\,n$	$\cdots$	n	0
index	$-kn$	$-(k-2)\,n$	$\cdots$	$(k-2)\,n$	kn

.

Example 8.3: Consider $D^2 : C^2_\delta \to C^0_{\delta-2}$. In particular, when our function spaces contain $\mathbb{R}^2$-valued functions (as in the Kepler problem in the plane), the kernel of D^2 (on C^2) consists of functions of the form $(m_1 t + c_1, m_2 t + c_2)$, a 4-dimensional function space. When $\delta \geq 1$, all of these functions belong to C^2_δ; but for $0 \leq \delta < 1$, only the constant functions (c_1, c_2) from the kernel of D^2 are in the function space C^2_δ. When $\delta < 0$, none of these functions survive in C^2_δ (except, of course, the trivial function $(0,0)$). So the dimension of the kernel of D^2 is 0 for $\delta < 0$, the dimension is 2 for $0 \leq \delta < 1$, and it is 4 for $\delta \geq 1$. As per the above tables, the dimensions of the cokernel are the same as the dimensions of the kernel, but written down in the opposite order. That gives us the following table, for the operator $D^2 : C^2_\delta \to C^0_{\delta-2}$ on spaces of $\mathbb{R}^2$-valued functions:

$D^2 : C^2_\delta \to C^0_{\delta-2}$	$\delta < 0$	$0 < \delta < 1$	$1 < \delta$
dim(kernel)	0	2	4
dim(cokernel)	4	2	0
index	-4	0	4

.

Lemma 8.3: Let $C^k_\delta \subset C^\ell_{\delta'}$, with $k > \ell$ and $\delta < \delta'$. Then the embedding $C^k_\delta \hookrightarrow C^\ell_{\delta'}$ (considered as a bounded linear operator) is a compact operator.

Proof: First, note that multiplication by $\langle t \rangle^{-\delta'-1}$ is a isomorphism that takes C^k_δ to $C^k_{\delta-\delta'-1}$ and takes $C^\ell_{\delta'}$ to C^ℓ_{-1}, so we can reduce the hypothesis to the case that $C^k_\delta \subset C^\ell_{-1}$ with $\delta < -1$. Next, the D^ℓ operator takes C^k_δ to $C^{k-\ell}_{\delta-\ell}$ and takes C^ℓ_{-1} to $C^0_{-1-\ell}$. The D^ℓ operator is not an isomorphism, but it is Fredholm (since δ and -1 are both negative and hence not indicial points). And in this case, D^ℓ is injective, so it possesses an exact left inverse (and a right inverse modulo a compact operator). A final multiplication by the isomorphism $\langle t \rangle^{\ell+1}$ reduces the hypothesis to the case that $C^{k-\ell}_\delta \subset C^0_0$ with $\delta < 0$. Furthermore, $C^{k-\ell}_\delta \subset C^1_\delta \subset C^0_0$, so it suffices to prove this lemma for the case that $C^1_\delta \subset C^0_0$ with $\delta < 0$.

Now let $\{\mathbf{s}_i\}_i$ be a bounded sequence in C^1_δ and let S be the closure in C^0_0 of the set $\{\mathbf{s}_i\}$. Since $\{\mathbf{s}_i\}$ is bounded in C^1_δ, there is a $c > 0$ such that, for all positive integers i and for all $t \in \mathbb{R}$ we have:

$$\|\mathbf{s}_i(t)\| \leq c \langle t \rangle^\delta, \tag{192}$$

$$\|\mathbf{s}'_i(t)\| \leq c \langle t \rangle^{\delta-1}. \tag{193}$$

In particular, since $\delta < 0$, both $\{\mathbf{s}_i\}$ and the derivatives $\{\mathbf{s}'_i\}$ are all bounded by a common constant. Applying the Mean Value Theorem to each component of $\mathbf{s}_i$ (in $\mathbb{R}^n$), we see that each component $\mathbf{s}_i$ satisfies a Lipschitz condition, with a Lipschitz constant that is independent of i. Hence the set $\{\mathbf{s}_i\}$ is uniformly bounded and equicontinuous. Now choose an $\varepsilon > 0$. From inequality (192) above, we know there is an $M > 0$ such that $\|\mathbf{s}_i(t)\| \leq \varepsilon$ for all t such that $|t| \geq M$. On the compact interval $[-M, M]$, the sequence $\{\mathbf{s}_i\}_i$ satisfies the hypotheses of the Ascoli-Arzela Theorem, hence it has a subsequence that converges uniformly to a continuous function $\mathbf{s}$ defined on $[-M, M]$. We can extend $\mathbf{s}$ to all of $\mathbb{R}$ by setting $\mathbf{s}(t) = \mathbf{s}(-M)$ for $t \in (-\infty, -M)$ and $\mathbf{s}(t) = \mathbf{s}(M)$ for $t \in (M, \infty)$. Clearly the extended $\mathbf{s}$ is in C_0^0 and $\|\mathbf{s}(t)\| \leq \varepsilon$ for all t such that $|t| \geq M$. Hence

$$\|\mathbf{s}_i(t) - \mathbf{s}(t)\| \leq \|\mathbf{s}_i(t)\| + \|\mathbf{s}(t)\| \leq 2\varepsilon \qquad (194)$$

for all i and for all t such that $|t| \geq M$; and by uniform convergence on $[-M, M]$, there is a positive integer N such that this also holds for all $i > N$ and for all t such that $|t| \leq M$. Thus, we have the fact that all but a finite number of elements of the set $\{\mathbf{s}_i\}$ are contained in a ball of radius 2ε about $\mathbf{s}$. The remaining elements $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N$ are trivially contained in balls of radius 2ε about themselves, so it follows that the set $\{\mathbf{s}_i\}$ is totally bounded.

The set S is totally bounded as well, because we can extend the balls to a radius of 3ε to cover any cluster points that are not already in $\{\mathbf{s}_i\}$ (since a cluster point must be within any ε of a member of $\{\mathbf{s}_i\}$). But the set S is complete, since it is a closed subset of a Banach space, and a set that is complete and totally bounded is compact. Therefore the sequence $\{\mathbf{s}_i\}_i$ lies in a compact set, so it must have a convergent subsequence that converges to a function in S and hence in C_0^0.

■

Corollary 8.4: Let $\varphi(t, \widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)$ be a hyperbolic solution to the Kepler problem. The linearized Kepler operator $D^2 - K_t : C^2_\delta \to C^0_{\delta-2}$, where

$$K_t = K(t) = \frac{M}{\|\mathbf{x}\|^5} \begin{bmatrix} 2x_1^2 - x_2^2 & 3x_1x_2 \\ \\ 3x_1x_2 & 2x_2^2 - x_1^2 \end{bmatrix}_{(x_1,x_2)=\varphi(t,\widetilde{x}_1,\widetilde{x}_2,\widetilde{y}_1,\widetilde{y}_2)},$$

is a Fredholm operator on the spaces of $\mathbb{R}^2$-valued functions $C^2_\delta \to C^0_{\delta-2}$, $\delta \in \mathbb{R}$, $\delta \neq 0$, $\delta \neq 1$. Furthermore, the index, and the dimensions of the kernel and cokernel, are given by the following table:

$D^2 - K_t : C^2_\delta \to C^0_{\delta-2}$	$\delta < 0$	$0 < \delta < 1$	$1 < \delta$
dim(kernel)	0	1	4
dim(cokernel)	4	1	0
index	-4	0	4

.

Proof: First, note that $C^2_{\delta-3} \subset C^0_{\delta-2}$ and by Lemma 8.3, the embedding $C^2_{\delta-3} \hookrightarrow C^0_{\delta-2}$ is compact (considered as a bounded linear operator). Since $K_t : C^2_\delta \to C^2_{\delta-3}$ is a bounded operator (in fact, $K_t \in C^\infty$ and it is bounded by $c/\langle t\rangle^3$, with $c > 0$), it follows that $K_t : C^2_\delta \to C^2_{\delta-3} \hookrightarrow C^0_{\delta-2}$ is a compact operator. Hence $D^2 - K_t : C^2_\delta \to C^0_{\delta-2}$ is a Fredholm operator because $D^2 : C^2_\delta \to C^0_{\delta-2}$ is Fredholm and K_t is a compact operator, and an operator that differs from a Fredholm operator by a compact operator is also Fredholm (see the Compact Perturbation Theorem, stated above). The dimensions of the kernel in the above table are an immediate corollary to Theorem 7.1. The index values follow from the Compact Perturbation Theorem, which states that two Fredholm operators that differ by a compact operator have the same index values, hence the index values of the linearized Kepler operator must agree with the known index values of the D^2 operator. The dimensions of the cokernel then follow by subtraction of the index values from the dimensions of the kernel.

■

Comment: The above results regarding the cokernel dimensions are confirmed by a tenet of Weighted Space Theory, which we describe here briefly and without elaboration. This tenet prescribes a relationship between the dimensions of the kernel and cokernel of a Fredholm operator that maps a continuum of δ-weighted spaces (such as the $C^2_\delta \to C^0_{\delta-2}$ spaces in the present instance), except for the indicial points (a discrete set of δ values). According to this tenet, the dimensions of the cokernel of a 2^{nd}-order operator, in each range of δ, bounded by indicial points, are the same as the dimensions of the kernel, but written down in the opposite order, reflected about the point $\delta = 1/2$.

9. Key Lemma

In this section, we denote the linearized Kepler operator by L. That is, $L = D^2 - K_t : C^2_\delta \to C^0_{\delta-2}$, where our function spaces contain $\mathbb{R}^2$-valued functions and

$$K_t = K(t) = \frac{M}{\|\mathbf{x}\|^5} \begin{bmatrix} 2x_1^2 - x_2^2 & 3x_1x_2 \\ & \\ 3x_1x_2 & 2x_2^2 - x_1^2 \end{bmatrix}_{(x_1,x_2)=\varphi(t,\widetilde{x}_1,\widetilde{x}_2,\widetilde{y}_1,\widetilde{y}_2)}, \tag{195}$$

where $\varphi(t, \widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)$ is a hyperbolic solution to the Kepler problem.

The objective of this research is to develop tools that can be used to show the existence of exact solutions to the three-body (and N-body) problem, subject to certain constraints. For instance, it is desirable to construct a family $\{\mathbf{x}(t)\}$ of approximate solutions, having prescribed hyperbolic escape trajectories, and then use the mapping properties of the linearized operator L to find correction terms $\mathbf{w}(t)$ so that $\mathbf{x}(t) + \mathbf{w}(t)$ are exact solutions. The main tool in such investigations is the Implicit Function Theorem (or, equivalently, the "Contraction Mapping Principle"), for which we require the surjectivity of L. But, as we have shown, L is not surjective for $\delta < 1$. This in turn would seem to force us to consider perturbations $\mathbf{w}(t) \in \mathcal{C}^2_\delta$ with $\delta > 1$, which leads to a seemingly untenable situation, since the growth of these correction terms could overwhelm the linear asymptotics of approximate solution.

To circumvent this impasse, we obtain a refinement of Corollary 8.4. This refinement shows that L is surjective on slightly modified function spaces. Suppose we have a given δ, with $0 < \delta < 1$. We choose another value, δ', such that $1 < \delta'$.

Ideally, we would like to find the correction term $\mathbf{w}(t)$ in $\mathcal{C}^2_\delta$ to solve away the error term $\mathbf{g} = \mathcal{N}(\mathbf{x}) \in \mathcal{C}^0_{-2} \subset \mathcal{C}^0_{\delta-2}$, where $\mathcal{N}$ is the non-linear Newtonian operator (of which L is the linearization). Since L is not surjective from $\mathcal{C}^2_\delta \to \mathcal{C}^0_{\delta-2}$, the best we can do is regard $\mathbf{g} \in \mathcal{C}^0_{\delta'-2}$, and find $\mathbf{f} \in \mathcal{C}^2_{\delta'}$ with $L(\mathbf{f}) = \mathbf{g}$. The point of this refinement is that since $\mathbf{g}$ decays more quickly than the general element of $\mathcal{C}^0_{\delta'-2}$, this solution $\mathbf{f}$ is correspondingly better than the general element of $\mathcal{C}^2_{\delta'}$.

Lemma 9.1: Let $\delta, \delta' \in \mathbb{R}$, with $0 < \delta < 1 < \delta'$. Fix $\mathbf{g} \in \mathcal{C}^0_{\delta-2}$ and select any $\mathbf{f} \in \mathcal{C}^2_{\delta'}$ with $L(\mathbf{f}) = \mathbf{g}$. Then for $t \geq 0$, there is a decomposition $\mathbf{f}(t) = \mathbf{h}^+(t) + \mathbf{v}^+ t$ where $\mathbf{h}^+ \in \mathcal{C}^2_\delta$ and $\mathbf{v}^+ \in \mathbb{R}^2$. Similarly, for $t \leq 0$, $\mathbf{f}(t) = \mathbf{h}^-(t) + \mathbf{v}^- t$ with $\mathbf{h}^- \in \mathcal{C}^2_\delta$ and $\mathbf{v}^- \in \mathbb{R}^2$.

Comments:

- Recall that a bounded linear operator is Fredholm if and only if it has left and right inverses modulo compact operators. We showed that L is Fredholm by a direct argument. However, a common method for showing that an operator is Fredholm is to construct "approximate left and right inverse" operators G_1, G_2 by using a partition of unity to paste together several "partial approximate inverses", which cover overlapping intervals of the variable t. Our proof of this lemma involves the construction of two such "partial approximate inverses".

- A key point is that since K decays only polynomially, it is necessary to think of L as having a regular singularity at ∞ and to use our monomially weighted spaces C^k_δ. Thus, we can write

$$L = t^{-2}\left((tD)^2 - tD - t^2K(t)\right) \tag{196}$$

on $(0,\infty)$; since $t^2K(t) = \mathcal{O}(1/t)$, the operator in parentheses on the right has a regular singularity at ∞. (See **[3]**.) Behavior of solutions is governed by the indicial roots, which in this case are equal to 0 and 1. These are the values s for which there exists $\mathbf{v} \in \mathbb{R}^2$ such that $t^{-(s-2)}L(t^s\mathbf{v}) \to 0$ as $t \to \infty$ (and in fact, for this particular operator and for $s = 0, 1$, this is true for any $\mathbf{v}$). By standard perturbation arguments, any solution of $L(\mathbf{f}) = 0$ satisfies

$$\mathbf{f}(t) = \mathbf{u} + \mathcal{O}((\log t)/t) \qquad \text{or} \qquad \mathbf{f}(t) = \mathbf{v}t + \mathcal{O}(\log t), \tag{197}$$

for some vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.

Proof: We focus on constructing G_+, a "partial approximate inverse" on the interval $I_+ = [0,\infty)$ (another "partial approximate inverse", G_- can be constructed on $I_- = (-\infty, 0]$ in essentially the same manner).

For any given $\delta \neq 0, 1$, we can choose $a, b \in [0,\infty]$ such that

$$(G_+(\mathbf{g}))(t) = \int_a^\tau \int_b^s \mathbf{g}(s)\, ds\, d\tau \tag{198}$$

defines an operator which is defined as a mapping between $\mathcal{C}^0_{\delta-2}(I_+)$ and $\mathcal{C}^2_\delta(I_+)$, and such that $D^2 \circ G_+ = I$. Recall from our proof that the operator D is Fredholm, that when both $\delta - 2$ and $\delta - 1$ are negative (in the case of the inner integral) or both $\delta - 1$ and δ are negative (in the case of the outer integral), we do not get an inverse valid on all of $\mathbb{R}$ unless a constraint of the form $\int_{-\infty}^{\infty} \mathbf{g} = 0$ is satisfied. However, the proof showed that, in such cases, the integrals of the form $\int_{\infty}^{s} \mathbf{g}$ do, in fact, provide an inverse on the interval $[0, \infty)$ (even though this partial inverse cannot, in general, be extended to $(-\infty, 0]$ unless the constraint holds).

In particular, when $\delta < 0$, let $a = b = \infty$; for $0 < \delta < 1$ (as is the case in this lemma), let $a = 0$, $b = \infty$; and for $\delta > 1$, let $a = b = 0$.

Now consider the case $t \geq 0$ in the statement of this lemma. By elementary calculus, the "partial approximate inverse" G_+ satisfies

$$\mathbf{f} - G_+ \circ D^2(\mathbf{f}) = \mathbf{w}, \tag{199}$$

$$G_+ \circ D^2(\mathbf{f}) = \mathbf{f} - \mathbf{w}, \tag{200}$$

where $\mathbf{w}(t) = \mathbf{a}^+ + \mathbf{v}^+ t$ is a linear function. Thus, since $L(\mathbf{f}) = \mathbf{g}$ can be written as $D^2(\mathbf{f}) = \mathbf{g} + K(\mathbf{f})$, we can apply G_+ to both sides and we obtain

$$G_+ \circ D^2(\mathbf{f}) = G_+(\mathbf{g}) + G_+ \circ K(\mathbf{f}), \tag{201}$$

$$\mathbf{f} - \mathbf{w} = G_+(\mathbf{g}) + G_+ \circ K(\mathbf{f}), \tag{202}$$

$$\mathbf{f} = G_+(\mathbf{g}) + G_+ \circ K(\mathbf{f}) + \mathbf{w}, \tag{203}$$

$$\mathbf{f}(t) = G_+(\mathbf{g})(t) + G_+ \circ K(\mathbf{f}(t)) + \mathbf{a}^+ + \mathbf{v}^+ t. \tag{204}$$

In equation (204) above, the function $G_+(\mathbf{g})$ lies in $\mathcal{C}^2_\delta$, and the constant term $\mathbf{a}^+$ also defines a function in $\mathcal{C}^2_\delta$ (since we are assuming that $\delta > 0$). Furthermore, note that multiplication of $\mathbf{f} \in \mathcal{C}^2_{\delta'}$ by the matrix K does not affect the differentiability of $\mathbf{f}$ but improves the "bounding envelope" from $\langle t \rangle^{\delta'}$ to $\langle t \rangle^{\delta'-3}$; and operation of G_+ on $K(\mathbf{f})$ (integration of $K(\mathbf{f})$ twice) improves the differentiability from C^2 to C^4 but degrades the "bounding envelope" from $\langle t \rangle^{\delta'-3}$ to $\langle t \rangle^{\delta'-1}$. Hence the second term, $G_+ \circ K(\mathbf{f})$, is in $\mathcal{C}^4_{\delta'-1}$. If $\delta' - 1 \leq \delta$, we can set $\mathbf{h}^+(t) = G_+(\mathbf{g})(t) + G_+ \circ K(\mathbf{f}(t)) + \mathbf{a}^+$ and we are done, because in that case $\mathbf{h}^+ \in \mathcal{C}^2_\delta$ and $\mathbf{f}(t) = \mathbf{h}^+(t) + \mathbf{v}^+ t$. If $\delta' - 1 > \delta$, then we can iterate this calculation a finite number of times to obtain the required decomposition.

The case $t \leq 0$ is proved similarly, using a "negative-axis partial approximate inverse" G_-.

∎

10. Conclusion

With Corollary 8.4 and Lemma 9.1, we have obtained results that characterize perturbation properties of the linearized Kepler operator. These properties are expected to be useful in analyzing the behavior, under perturbations, of approximate solutions to the three-body (and N-body) problem. In particular, we are interested in future research that applies these results to N-body systems that satisfy the following definition, which is based on the Chazy Theorem that was presented in our Introduction:

Definition: Given a Newtonian gravitational system of N bodies with masses $m_1, m_2, \ldots, m_N$, with $\mathbb{R}^d$ as the coordinate space of each body. We say that a solution $\mathbf{x}(t) \in \mathbb{R}^{dN}$ exhibits *hyperbolic escape in the future* if each body's velocity $\dot{\mathbf{x}}_i(t)$ tends to a non-zero limit as $t \to +\infty$, and simultaneously, the distances $r_{ij}(t) = \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|$ between any two of the bodies tends to infinity. One defines *hyperbolic escape in the past* analogously.

Comment: Recall from the Chazy Theorem that any solution with future hyperbolic escape has an asymptotic expansion of the form

$$\mathbf{x}_i(t) = \mathbf{A}_i^+ \, t + \mathbf{B}_i^+ \log t + \mathbf{C}_i^+ + \mathcal{O}\left(\frac{\log t}{t}\right), \tag{205}$$

where $\mathbf{A}_i^+, \mathbf{B}_i^+, \mathbf{C}_i^+$ are constant vectors in $\mathbb{R}^d$. Similarly, any solution with past hyperbolic escape has an asymptotic expansion of the form

$$\mathbf{x}_i(t) = \mathbf{A}_i^- \, t + \mathbf{B}_i^- \log t + \mathbf{C}_i^- + \mathcal{O}\left(\frac{\log t}{t}\right), \tag{206}$$

where $\mathbf{A}_i^-, \mathbf{B}_i^-, \mathbf{C}_i^-$ are constant vectors in $\mathbb{R}^d$.

Definitions: Let $\mathcal{H}$ denote the set of all solutions $\mathbf{x}(t)$ of the Newton problem that have both past and future hyperbolic escape. We call these simply *hyperbolic escape orbits*. Associated to any $\mathbf{x} \in \mathcal{H}$ are two N-tuples of vectors

$$\mathbf{A}^- = (\mathbf{A}_1^-, \ldots, \mathbf{A}_N^-), \qquad \text{and} \qquad \mathbf{A}^+ = (\mathbf{A}_1^+, \ldots, \mathbf{A}_N^+),$$

determined by $\mathbf{x}_i(t) \sim \mathbf{A}_i^{\pm} t$ as $t \to \pm\infty$. (Note that by this convention, the incoming asymptotic ray is the one through $-A_i^-$.) We define the *end-map* $\mathbf{E} : \mathcal{H} \to \mathbb{R}^{dN} \times \mathbb{R}^{dN}$ by:

$$\mathbf{E}(\mathbf{x}) = (\mathbf{A}^-, \mathbf{A}^+) \,.$$

Some of our questions for future research can then be phrased as asking about properties of this mapping, in particular, about its range, injectivity, *etc.* In particular, we hope to be able to prove theorems along the line of the following:

Conjecture: Given a Newtonian gravitational system of N bodies with masses $m_1, m_2, \ldots, m_N$, with $\mathbb{R}^d$ as the coordinate space of each body. Let $(\mathbf{A}^-, \mathbf{A}^+) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$ be such that

i) $\mathbf{A}_j^- \wedge \mathbf{A}_j^+ \neq 0$, $j = 1, \ldots, N$;

ii) $\sum_{j=1}^N m_j \mathbf{A}_j^- = \sum_{j=1}^N m_j \mathbf{A}_j^+$;

iii) $\sum_{j=1}^N m_j \left\|\mathbf{A}_j^-\right\|^2 = \sum_{j=1}^N m_j \left\|\mathbf{A}_j^+\right\|^2$.

Then there exists an $(N-1)$-dimensional family of collision-free hyperbolic escape orbits $\mathbf{x}(t) \in \mathcal{H}$ such that their end maps are $\mathbf{E}(\mathbf{x})$ are the prescribed vectors $(\mathbf{A}^-, \mathbf{A}^+)$. These solutions are "weakly coupled", in the sense that for any three distinct indices i, j, k, the longest side of the triangle in $\mathbb{R}^d$ with vertices $\mathbf{x}_i(t), \mathbf{x}_j(t), \mathbf{x}_k(t)$ is uniformly large for all time.

Note: It is possible that we may have to weaken the conclusion of this conjecture somewhat. It may turn out that we can show the existence of solutions that match the prescribed incoming or outgoing asymptotes exactly, but not both. For instance, we may find that, for any $\varepsilon > 0$, there exists a solution $\mathbf{x}(t) \in \mathcal{H}$ with end map $\mathbf{E}(\mathbf{x}) = (\mathbf{A}^-, \mathbf{A}^+ + \mathbf{v})$, such that $\|\mathbf{v}\| < \varepsilon$.

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